On Transformation Optics based Cloaking

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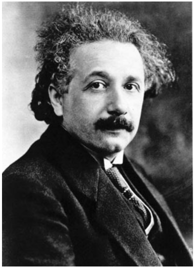
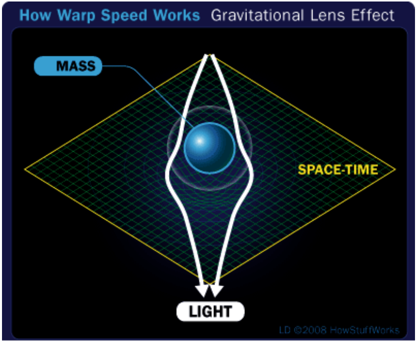
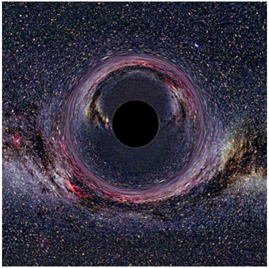
Outline

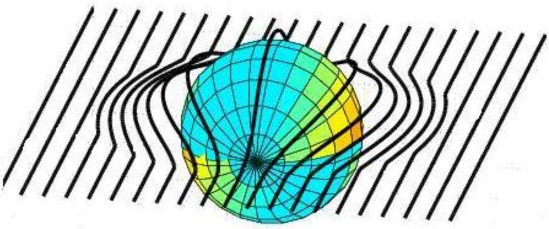
Deflecting light



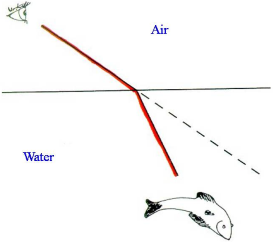


Black holes and optical white holes



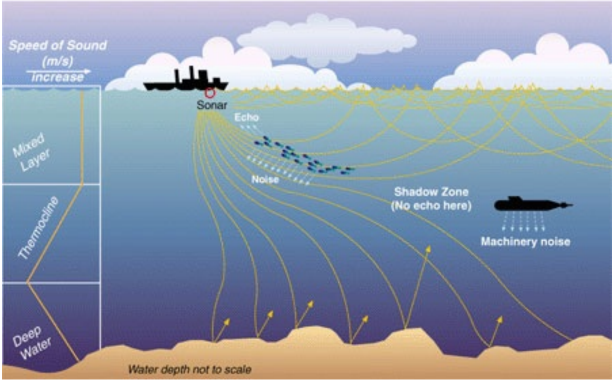


Mirage

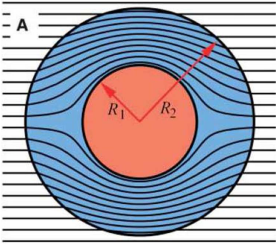


Fermat's principle: *minimize optical length in a medium with variable refractive index.*

Cloaking for acoustics



Transformation Optics based Cloaking



From Pendry et al's paper

- J. B. Pendry, D. Schurig and D. R. Smith (2006)
- U. Leonhard (2006)
- Transformation Optics and *Metamaterials*
- A. Greenleaf, M. Lassas and G. Uhlmann (2003)

Inverse Problem and Invisibility

Visibility: Inverse problems of EIT (Calderón problem)

Electrical Impedance Tomography (EIT)

- Conductivity equation:

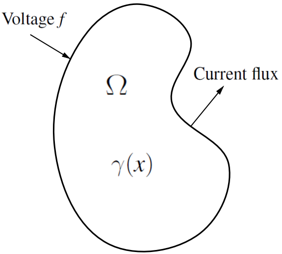
$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega.$$

$$(0 < C_1 \leq \gamma \leq C_2)$$

- Boundary measurements:

Dirichlet-to-Neumann (DN) map

$$\Lambda_\gamma : u|_{\partial\Omega} \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}$$



Calderón problem: $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$?

- Isotropic γ scalar: uniqueness results —- Visibility [Sylvester-Uhlmann, Nachman, . . .]
- Anisotropic $\gamma = (\gamma^{ij})$ (pos. def. sym. tensor): non-uniqueness.

Transformation law and nonuniqueness

$$\int_\Omega \nabla u_f \cdot \gamma \nabla u_f \, dy \overset{x=\psi(y),\psi|_{\partial\Omega}=I}{=} \int_\Omega \nabla v_f \cdot \underbrace{\left(\frac{(D\psi)^T \gamma \, (D\psi)}{|\det(D\psi)|}\right) \circ \psi^{-1}}_{\psi_* \gamma} \nabla v_f \, dx$$

$$\int_{\partial\Omega} \Lambda_\gamma(f) f \, dS_y = \int_{\partial\Omega} \Lambda_{\psi_*\gamma}(f) f \, dS_x$$

DN-map:

$$\begin{cases} \nabla \cdot \gamma \nabla u_f = 0 \\ u_f|_{\partial\Omega} = f \\ \boxed{\Lambda_\gamma(f) = \nu \cdot \gamma \nabla u_f|_{\partial\Omega}} \end{cases} \overset{v_f = u_f \circ \psi^{-1}}{\Longrightarrow} \begin{cases} \nabla \cdot \psi_*\gamma \nabla v_f = 0 \\ v_f|_{\partial\Omega} = f \\ \boxed{\Lambda_{\psi_*\gamma}(f) = \nu \cdot \psi_*\gamma \nabla v_f|_{\partial\Omega}} \end{cases}$$

Conclusion: ψ: a diffeomorphism on Ω and $\psi|\partial\Omega = Id$.

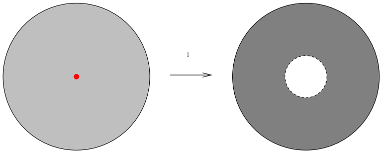
$$\boxed{\Lambda_\gamma = \Lambda_{\psi_*\gamma}}$$

Cloaking for EIT

$$F : B_2 \backslash \{0\} \to B_2 \backslash \overline{B_1}$$

$$F(y) = \left(1 + \frac{|y|}{2}\right) \frac{y}{|y|}.$$

$$F|_{\partial B_2} = \text{Identity}.$$



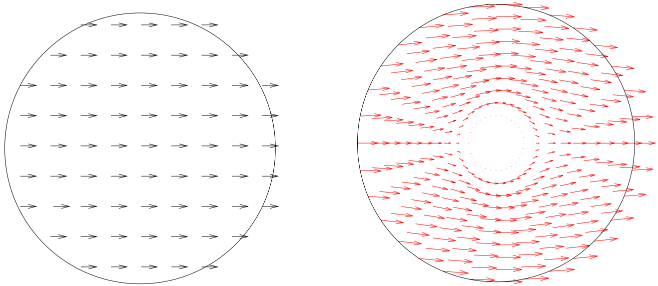
Greenleaf-Lassas-Uhlmann (2003)

$$\left.\begin{array}{l} \gamma = I : \text{Identity matrix in } B_2, \\ \tilde{\gamma} = \begin{cases} F_*\gamma & \text{in } B_2 \backslash \overline{B_1} \\ \text{arbitrary } \gamma_a & \text{in } B_1 \end{cases} \end{array}\right\} \Rightarrow \boxed{\Lambda_{\tilde{\gamma}} = \Lambda_\gamma}.$$

- F_*I is anisotropic.
- Removable singularity argument.

Currents (vacuum space vs. cloaking)

All Boundary measurements for the homogeneous conductivity $\gamma = I$ and the conductivity $\tilde{\gamma} = (F_*I, \gamma_a)$ are the same



Analytic solutions for the currents

Based on work of Greenleaf-Lassas-Uhlmann, 2003

Singular Ideal Electromagnetic Cloaking

Wave theory of light: Electromagnetic waves and Maxwell's equations

$$\boxed{\begin{array}{l} \nabla \times E - i\omega\mu H = 0 \\ \nabla \times H + i\omega\varepsilon E = J \end{array}}$$

(E, H): electromagnetic field

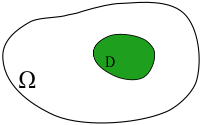
$\mu(x)$: magnetic permeability

$\varepsilon(x)$: electric permittivity

$J(x)$: electric current source

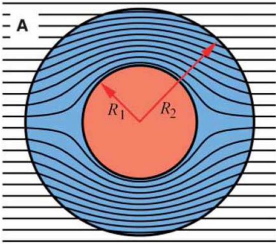
Refractive index: $\sqrt{\mu\varepsilon}$.

What is invisibility?



Arbitrary object to be cloaked in D surrounded by the cloak $\Omega\backslash\overline{D}$ with electromagnetic parameters $(\tilde{\mu}(x), \tilde{\varepsilon}(x))$. We want to show that if Maxwell's equations are solved in Ω, the boundary information of solutions is the same as that of the case with $\mu = \varepsilon = Id$.

Invisibility and Cloaking



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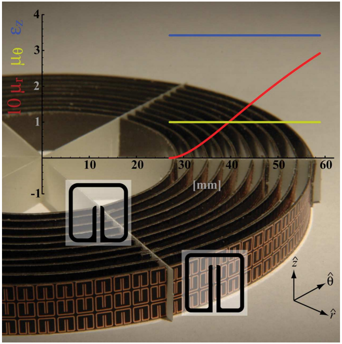
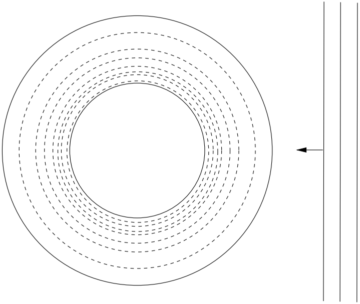
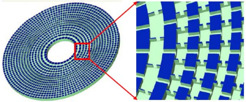
Metamaterials for electromagnetic cloaking



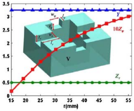


Invisibility cloak for 4 cm EM waves
Schurig et al, Science 2006.

Metamaterials for acoustic cloaking



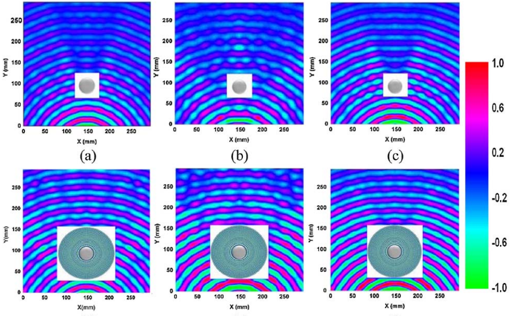
(a)



(b)

Layer	l_r(mm)	l_φ(mm)	V(mm^3)
1	2.05	0.10	3.00
3	1.37	0.22	2.29
5	1.24	0.41	2.06
7	1.24	0.30	2.06
9	1.24	0.41	2.06
11	1.24	0.52	2.06
13	1.24	0.63	2.06
15	1.24	0.74	2.06

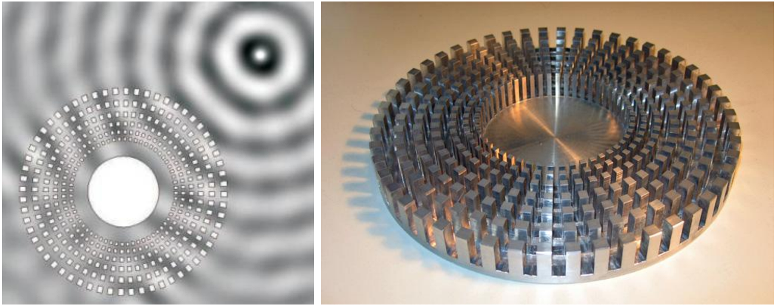
(c)



(a) (b) (c) (d) (e) (f)

Zhang et al, PRL 2011

Tsunami cloaking



Broadband cylindrical cloak for linear surface waves in a fluid,
M. Farhat et al, PRL (2008).

Electromagnetic waves in regular media

- Time harmonic Maxwell's equations

$$\nabla \times E = i\omega\mu H \quad \nabla \times H = -i\omega\varepsilon E + J \quad \text{in } \Omega.$$

  with permittivity $\varepsilon(x)$ and permeability $\mu(x)$.
- **Regular (Nonsingular)** medium: $\varepsilon = (\varepsilon^{ij})$ and $\mu = (\mu^{ij})$ are pos. def. sym. matrices, that is, there exists $C > 0$ such that

$$\sum_{i,j} \mu^{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad \sum_{i,j} \varepsilon^{ij}(x)\xi_i\xi_j \geq C|\xi|^2$$

  for $\xi \in \mathbb{R}^n$ and $x \in \Omega$.
- Then $(E, H) \in H(\text{curl}) \times H(\text{curl})$.

Imaging and inverse problems with electromagnetic waves

- Boundary observation: **Impedance map**

$$\Lambda_{\mu,\varepsilon} : \nu \times E|_{\partial\Omega} \mapsto \nu \times H|_{\partial\Omega}.$$

- Inverse problem: Is $(\mu, \varepsilon) \mapsto \Lambda_{\mu,\varepsilon}$ injective?
  [Ola-Päivärinta-Somersalo], [Ola-Somersalo]: C^2 isotropic.

Transformation law for Maxwell's equations

Let $\psi : \Omega \to \Omega$ be a diffeomorphism.

- Pullback of fields by ψ^{-1}:

$$\tilde{E} = (\psi^{-1})^* E := (D\psi^T)^{-1} E \circ \psi^{-1}$$

$$\tilde{H} = (\psi^{-1})^* H := (D\psi^T)^{-1} H \circ \psi^{-1}$$

$$\tilde{J} = (\psi^{-1})^* J := [\det(D\psi)]^{-1} D\psi\, J \circ \psi^{-1}$$

- Push-forward of medium by ψ:

$$\tilde{\mu} = \psi_* \mu := \left(\frac{(D\psi)^T \mu\, (D\psi)}{|\det(D\psi)|} \right) \circ \psi^{-1},$$

$$\tilde{\varepsilon} = \psi_* \varepsilon := \left(\frac{(D\psi)^T \varepsilon\, (D\psi)}{|\det(D\psi)|} \right) \circ \psi^{-1}.$$

- Then

$$\boxed{\nabla \times \tilde{E} = i\omega \tilde{\mu} \tilde{H}, \quad \nabla \times \tilde{H} = -i\omega \tilde{\varepsilon} \tilde{E} + \tilde{J}} \quad \text{in } \Omega$$

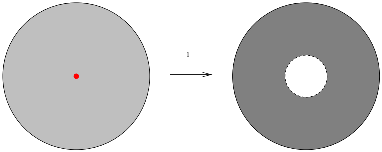
- Moreover, if $\psi|_{\partial\Omega} =$ Identity, we have $\boxed{\Lambda_{\tilde{\mu},\tilde{\varepsilon}} = \Lambda_{\mu,\varepsilon}}$

Electromagnetic cloaking medium

$$F: B_2\backslash\{0\} \to B_2\backslash\overline{B_1}$$

$$F(y) = \left(1 + \frac{|y|}{2}\right)\frac{y}{|y|}.$$

$$F|_{\partial B_2} = \text{Identity}.$$

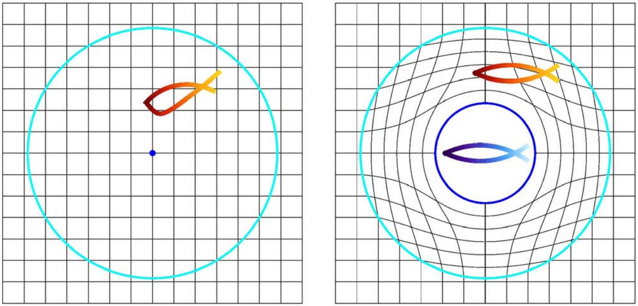


Cloaking medium

$$(\tilde{\mu}, \tilde{\varepsilon}) = \begin{cases} (F_*I, F_*I) & \text{in } B_2\backslash\overline{B_1} \\ (\mu_a, \varepsilon_a) \text{ arbitrary} & \text{in } B_1 \end{cases}$$

- Heterogeneous, anisotropic and singular in the cloaking layer.

Transformation Optics for Rays



Singular cloaking medium

- 3D cloaking device medium in $B_2 \backslash \overline{B}_1$:

$$\widetilde{\mu} = \widetilde{\varepsilon} = F_* I = \boxed{2\frac{(|x|-1)^2}{|x|^2}\Pi(x)} + 2(I - \Pi(x))$$

where $\Pi(x) = \hat{x}\hat{x}^T = xx^T/|x|^2$ is the projection along the radial direction.

- Degenerate singularity at $|x| = 1^+$!
- v.s. *Non-singular (Regular) medium*: for some $C > 0$,

$$\sum_{i,j} \gamma^{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad \xi \in \mathbb{R}^n,\ x \in \Omega$$

Finite energy solutions [Greenleaf-Kurylev-Lassas-Uhlmann]

Finite energy solution (FES) to Maxwell's equations for $(B_2, \tilde{\mu}, \tilde{\varepsilon})$:

$\tilde{E}, \tilde{H}, \tilde{D} = \tilde{\varepsilon}\tilde{E}$ and $\tilde{B} = \tilde{\mu}\tilde{H}$ are forms in B_2 with $L^1(B_2, dx)$-coefficients such that

$$\int_{B_2} \tilde{\varepsilon}^{ij}\, \tilde{E}_i\, \overline{\tilde{E}_j}\, dx < \infty, \qquad \int_{B_2} \tilde{\mu}^{ij}\, \tilde{H}_i\, \overline{\tilde{H}_j}\, dx < \infty,$$

Maxwell's equations hold in a neighborhood of ∂B_2, and

$$\int_{B_2} (\nabla \times \tilde{h}) \cdot \tilde{E} - \tilde{h} \cdot i\omega\tilde{\mu}\tilde{H}\, dx = 0$$

$$\int_{B_2} (\nabla \times \tilde{e}) \cdot \tilde{H} + \tilde{e} \cdot (i\omega\tilde{\varepsilon}\tilde{E} - \tilde{J})\, dx = 0$$

for all $\tilde{e}, \tilde{h} \in C_0^\infty(B_2)$.

- *Hidden boundary condition*: $\nu \times \tilde{E}|_{\partial B_1^-} = \nu \times \tilde{H}|_{\partial B_1^-} = 0$.
- Then **cloaking a source** ($\tilde{J}|_{B_1} \neq 0$) is problematic!

Regularized Electromagnetic Approximate Cloaking

Blow-up-a-small-ball regularization

- **Regularized** transformation that blows up $B_\rho (0 < \rho < 1)$ to B_1 and fixes the boundary ∂B_2.

$$F_\rho(y) := \begin{cases} \left(\frac{2(1-\rho)}{2-\rho} + \frac{|y|}{2-\rho}\right)\frac{y}{|y|}, & \rho < |y| < 2, \\ \frac{y}{\rho}, & |y| < \rho. \end{cases}$$

- Construct **non-singular** EM *anisotropic* material

$$(\tilde{\mu}_\rho, \tilde{\varepsilon}_\rho) := \begin{cases} ((F_\rho)_* I, (F_\rho)_* I), & 1 < |x| < 2, \\ (\mu_0, \varepsilon_0), & |x| < 1. \end{cases}$$

- Blow-up-a-small-ball regularization scheme for Helmholtz equations [Kohn-Onofrei-Vogelius-Weinstein]

Regularized cloaking medium

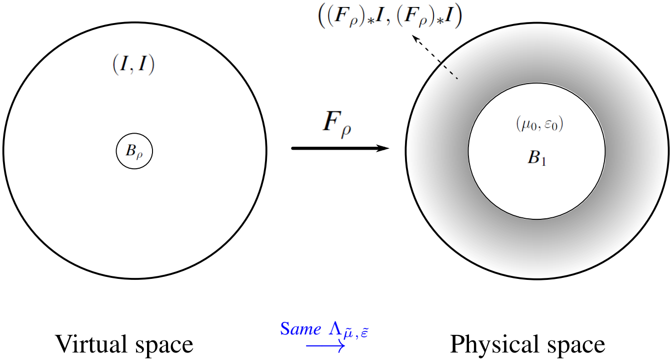
- Construct **non-singular** EM *anisotropic* material

$$(\tilde{\mu}_\rho, \tilde{\varepsilon}_\rho) := \begin{cases} ((F_\rho)_* I, (F_\rho)_* I), & 1 < |x| < 2, \\ (\mu_0, \varepsilon_0), & |x| < 1. \end{cases}$$

- $$(F_\rho)_* I = \frac{\big((2-\rho)|x| - 2 + 2\rho\big)^2}{(2-\rho)|x|^2} \Pi(x) + (2-\rho)(I - \Pi(x))$$

- Well-posedness: well-defined $H(\text{curl})$ solutions satisfying transmission problems in both physical space (cloaking device + cloaked region) and virtual space (pullback of physical space).

Virtual space vs. Physical space



- Is $\Lambda_{\tilde{\mu},\tilde{\varepsilon}} \approx \Lambda_{I,I}$? Yes and No!
- What is the limiting behavior (as $\rho \to 0$) of the EM waves in the physical space *at the interface* $|x| = 1$?

Transmission problems in physical and virtual spaces

Virtual space:

for $y \in B_2 \backslash \overline{B_\rho}$:

$$\nabla \times E_\rho^+ = i\omega H_\rho^+$$

$$\nabla \times H_\rho^+ = -i\omega E_\rho^+ + J$$

for $y \in B_\rho$:

$$\nabla \times E_\rho^- = i\omega((F_\rho^{-1})_* \mu_0) H_\rho^-$$

$$\nabla \times H_\rho^- = -i\omega((F_\rho^{-1})_* \varepsilon_0) E_\rho^- + J$$

$$\nu \times E_\rho^+|_{\partial B_\rho^+} = \nu \times E_\rho^-|_{\partial B_\rho^-}$$

$$\nu \times H_\rho^+|_{\partial B_\rho^+} = \nu \times H_\rho^-|_{\partial B_\rho^-}$$

$$\nu \times E_\rho^+|_{\partial B_2} = f$$

Physical space:

for $x \in B_2 \backslash \overline{B_1}$:

$$\nabla \times \tilde{E}_\rho^+ = i\omega \tilde{\mu}_\rho \tilde{H}_\rho^+$$

$$\nabla \times \tilde{H}_\rho^+ = -i\omega \tilde{\varepsilon}_\rho \tilde{E}_\rho^+ + \tilde{J},$$

for $x \in B_1$:

$$\nabla \times \tilde{E}_\rho^- = i\omega \mu_0 \tilde{H}_\rho^-$$

$$\nabla \times \tilde{H}_\rho^- = -i\omega \varepsilon_0 \tilde{E}_\rho^- + \tilde{J}$$

$$\nu \times \tilde{E}_\rho^+|_{\partial B_1^+} = \nu \times \tilde{E}_\rho^-|_{\partial B_1^-},$$

$$\nu \times \tilde{H}_\rho^+|_{\partial B_1^+} = \nu \times \tilde{H}_\rho^-|_{\partial B_1^-},$$

$$\nu \times \tilde{E}_\rho^+|_{\partial B_2} = f.$$

Cloaking a passive medium: $\tilde{J} = 0$

Assume μ_0 and ε_0 are positive constants, $k = \sqrt{\mu_0 \varepsilon_0}$.

- Spherical expansion of E's:

$$\tilde{E}_\rho^- = \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \alpha_n^m M_{n,k\omega}^m + \beta_n^m \nabla \times M_{n,k\omega}^m \quad \text{in } B_1,$$

$$E_\rho^+ = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_n^m N_{n,\omega}^m + d_n^m \nabla \times N_{n,\omega}^m + \gamma_n^m M_{n,\omega}^m + \eta_n^m \nabla \times M_{n,\omega}^m \quad \text{in } B_2 \backslash \overline{B_\rho}.$$

- Expansion of H's:

$$\tilde{H}_\rho^- = \frac{1}{ik\omega} \mu_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} k^2 \omega^2 \beta_n^m M_{n,k\omega}^m + \alpha_n^m \nabla \times M_{n,k\omega}^m \quad \text{in } B_1,$$

$$H_\rho^+ = \frac{1}{i\omega} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \omega^2 d_n^m N_{n,\omega}^m + c_n^m \nabla \times N_{n,\omega}^m + \omega^2 \eta_n^m M_{n,\omega}^m + \gamma_n^m \nabla \times M_{n,\omega}^m,$$

Cloaking a passive medium: $\tilde{J} = 0$

Plug in to the boundary conditions and transmission conditions:

$$\begin{cases} c_n^m h_n^{(1)}(2\omega) + \gamma_n^m j_n(2\omega) = f_{nm}^{(1)}, \\ d_n^m \mathcal{H}_n(2\omega) + \eta_n^m \mathcal{J}_n(2\omega) = 2f_{nm}^{(2)}. \end{cases}$$

$$\begin{cases} \rho c_n^m h_n^{(1)}(\omega\rho) + \rho\gamma_n^m j_n(\omega\rho) = \varepsilon_0^{-1/2} \alpha_n^m j_n(k\omega), \\ d_n^m \mathcal{H}_n(\omega\rho) + \eta_n^m \mathcal{J}_n(\omega\rho) = \varepsilon_0^{-1/2} \beta_n^m \mathcal{J}_n(k\omega). \end{cases}$$

$$\begin{cases} kc_n^m \mathcal{H}_n(\omega\rho) + k\gamma_n^m \mathcal{J}_n(\omega\rho) = \mu_0^{-1/2} \alpha_n^m \mathcal{J}_n(k\omega), \\ \rho d_n^m h_n^{(1)}(\omega\rho) + \rho\eta_n^m j_n(\omega\rho) = \mu_0^{-1/2} k\beta_n^m j_n(k\omega). \end{cases}$$

- Systems of linear equations for $X = (\alpha_n^m, \beta_n^m, c_n^m, d_n^m, \gamma_n^m, \eta_n^m)$

$$\mathcal{A}_{n,\rho,\omega,\mu_0,\varepsilon_0} X = b_f$$

Cloaking a passive medium: $\tilde{J} = 0$

Convergence order as $\rho \to 0$:

$$\gamma_n^m = O(1),\, \eta_n^m = O(1);\; c_n^m = O(\rho^{2n+1}),\; d_n^m = O(\rho^{2n+1});$$
$$\alpha_n^m = O(\rho^{n+1}),\; \beta_n^m = O(\rho^{n+1}).$$

$$\Lambda_{\tilde{\mu}_\rho,\tilde{\varepsilon}_\rho} \to \Lambda_{I,I}$$

Inside B_1,

$$(\tilde{E}_\rho^-, \tilde{H}_\rho^-) \to 0$$

Cloaking a medium with a source: $\tilde{J} \neq 0$ supported in B_1

Given an internal current source $\tilde{J}$ supported in B_{r_1} where $r_1 < 1$,

- Spherical expansion:

$$\tilde{E}_\rho^- = \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \alpha_n^m M_{n,k\omega}^m + \beta_n^m \nabla \times M_{n,k\omega}^m + p_n^m N_{n,k\omega}^m + q_n^m \nabla \times N_{n,k\omega}^m$$

for $\underline{r_1 < |x| < 1}$.

$$E_\rho^+ = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_n^m N_{n,\omega}^m + d_n^m \nabla \times N_{n,\omega}^m + \gamma_n^m M_{n,\omega}^m + \eta_n^m \nabla \times M_{n,\omega}^m$$

for $\underline{\rho < |y| < 2}$.

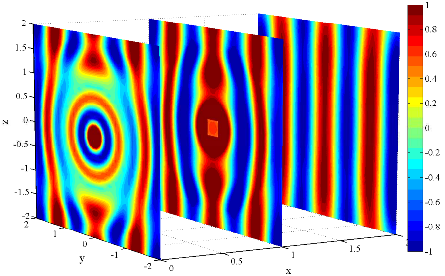
Cloaking a medium with a source: $\tilde{J} \neq 0$ supported in B_1

Convergence order as $\rho \to 0$:

$$\boxed{\begin{gathered}\gamma_n^m = O(1),\ \eta_n^m = O(1);\ c_n^m = O(\rho^{n+1}),\ d_n^m = O(\rho^{n+1});\\ \alpha_n^m = O(1),\ \beta_n^m = O(1).\end{gathered}}$$

$$\boxed{\Lambda_{\tilde{\mu}_\rho, \tilde{\varepsilon}_\rho} \to \Lambda_{I,I}}$$

Demonstration (passive)

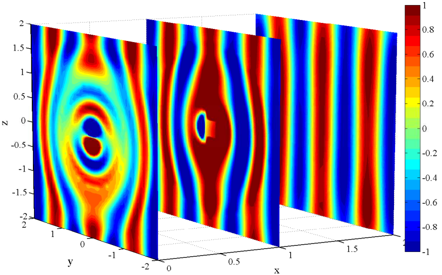


$\mathrm{Re}(\tilde{E}_\rho)_1$ (sliced at $x = 0, 1, 2$), $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, $\rho = 1/6$.

ρ	0.1	0.05	0.01	0.005	0.002	0.001
$Er(\rho)$	0.1810	0.0139	$8.42e-05$	$1.02e-06$	$6.42e-07$	$7.97e-08$
$r(\rho)$		3.703	3.173	3.044	3.020	3.009

Boundary errors and convergence order when $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$.

Demonstration (active)



$\mathrm{Re}(\tilde{E}_\rho)_1$ (sliced at $x = 0, 1, 2$), $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, $\rho = 1/12$, with a point source.

ρ	0.1	0.05	0.01	0.005	0.002	0.001
$Er(\rho)$	1.9787	0.3509	0.0114	0.0028	$4.41e-04$	$1.10e-04$
$r(\rho)$		2.495	2.129	2.031	2.013	2.006

Boundary errors and convergence order $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, with a point source.

Resonance

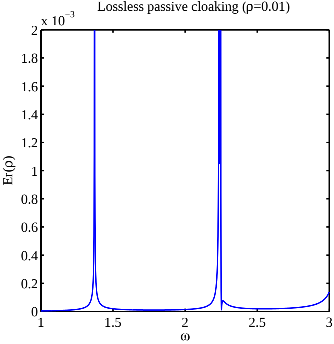
Resonance and Cloak-busting inclusions

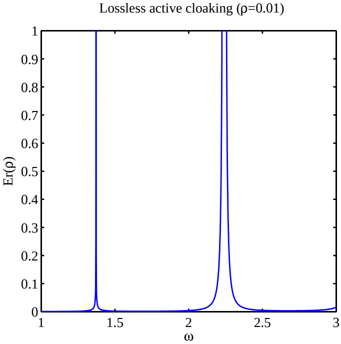
- For a fixed cloaking scheme, i.e., fixed $\rho > 0$, there exists some frequency ω and cloaked medium (μ_0, ε_0) such that the transmission problems are **NOT well-posed**. Therefore, the boundary measurement $\Lambda_{\tilde{\mu},\tilde{\varepsilon}}$ is significantly different from $\Lambda_{I,I}$.

$$\mu_0^{-1/2} \rho h_n^{(1)}(\omega\rho)\mathcal{J}_n(k\omega) - \varepsilon_0^{-1/2} k\mathcal{H}_n(\omega\rho) j_n(k\omega) = 0$$

- ω is the resonant frequency;
- (μ_0, ε_0) is called cloak-busting inclusion;

Demonstration (Resonance)



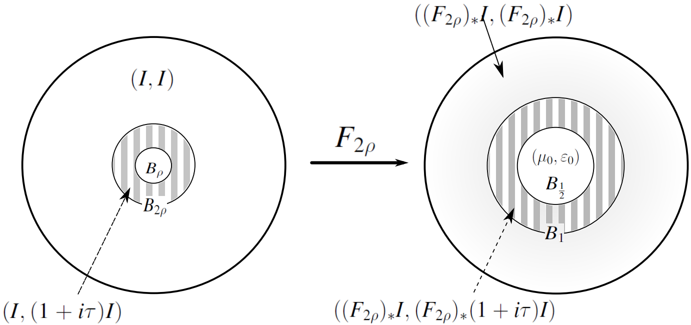


Boundary error $Er(\omega) = \nu \times H_\rho^+|_{\partial B_2} - \nu \times H|_{\partial B_2}$ for mode $n = 1$, when $\rho = 0.01$ and $\mu_0 = \varepsilon_0 = 2$, against frequency $\omega \in [1, 3]$ (Left: passive; Right: active).

Remedy to resonance: Cloaking with a lossy layer [1].

[1] [Kohn-Onofrei-Vogelius-Weinstein] for Helmholtz equations

Remedy: Lossy layer

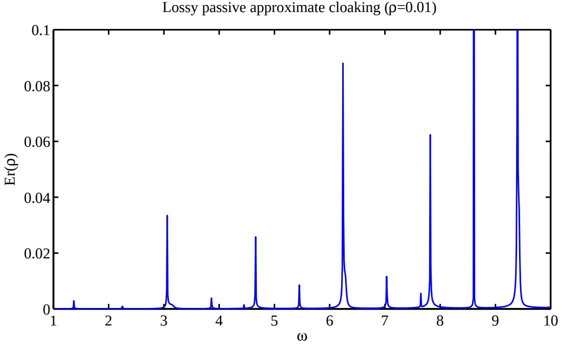


- $F_{2\rho}$ blows up $B_{2\rho}$ to B_1,

$$F_{2\rho}(y) := \begin{cases} \left(\frac{2(1-2\rho)}{2-2\rho} + \frac{|y|}{2-2\rho}\right)\frac{y}{|y|}, & 2\rho < |y| < 2, \\ \frac{y}{2\rho}, & |y| < 2\rho. \end{cases}$$

- τ is the damping parameter (conductivity).

Demonstration (lossy)



Boundary error *Er* for mode $n = 1$ when $\rho = 0.01$ of lossy approximate cloaking (passive), against frequency $\omega \in [1, 10]$.

- *Resonant frequencies disappear.*
- *Complex poles?*
- *Damping effect: τ depending on ρ.*

Damping parameter τ

- Lossy regularization for Scalar optics and Acoustics (Helmholtz equations) (Kohn-Onofrei-Vogelius-Weinstein, Kohn-Nguyen).

$$\nabla \cdot \gamma \nabla u + k^2 q u = 0$$

lossy cloaking medium

$$(\tilde{\gamma}_\rho, \tilde{q}_\rho) = \begin{cases} ((F_{2\rho})_* I, (F_{2\rho})_* 1) & 1 < |x| < 2 \\ ((F_{2\rho})_* I, (F_{2\rho})_* (1 + i c_0 \rho^{-2})) & 1/2 < |x| < 1 \\ (\gamma_0, q_0) & |x| < 1/2 \end{cases}$$

where

$$F_* q := \frac{q}{\det(DF)} \circ F^{-1}$$

Then

$$\|\Lambda_{\tilde{\gamma}_\rho, \tilde{q}_\rho} - \Lambda_{I,1}\| \lesssim \begin{cases} |\ln \rho|^{-1}, & \text{in } \mathbb{R}^2 \\ \rho, & \text{in } \mathbb{R}^3. \end{cases}$$

Extreme case: Enhanced cloaking by lining

- $\tau = \infty \;\Rightarrow$ sound-soft lining [Liu]. Then

$$\boxed{\|\Lambda_{\tilde{\gamma}_\rho,\tilde{q}_\rho} - \Lambda_{I,1}\| \lesssim \begin{cases} |\ln \rho|^{-1}, & \text{in } \mathbb{R}^2 \\ \rho, & \text{in } \mathbb{R}^3. \end{cases}}$$

(High loss makes detection easier in infrared regime!)

- Finite-sound-hard layer [Liu]:

$$(\tilde{\gamma}_\rho, \tilde{q}_\rho) = \begin{cases} ((F_{2\rho})_* I, (F_{2\rho})_* 1) & 1 < |x| < 2 \\ (\rho^{-2-\delta}(F_{2\rho})_* I, (F_{2\rho})_*(\alpha + i\beta)) & 1/2 < |x| < 1 \\ (\gamma_0, q_0) & |x| < 1/2 \end{cases}$$

Then

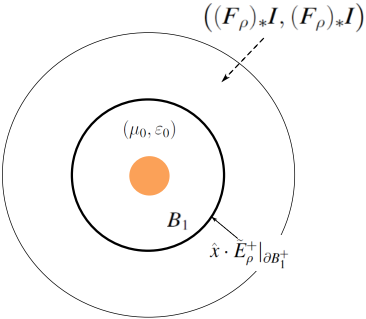
$$\boxed{\|\Lambda_{\tilde{\gamma}_\rho,\tilde{q}_\rho} - \Lambda_{I,1}\| \lesssim \rho^n} \quad \text{in } \mathbb{R}^n.$$

- In FSH, let $\delta \to \infty$, we have sound-hard lining.

Normal limits at the interface due to an internal source[2]

[2]This is a joint work with Prof. Matti Lassas

Radiation at the interface due to the internal source



- Given a current source $\tilde{J}$ supported on B_{r_1} for $r_1 < 1$. No resonance.
- As $\rho \to 0$, degenerate singularity arises at ∂B_1^+.
- Consider the limit of $\boxed{\hat{x} \cdot \tilde{E}_\rho^+}$ as $\rho \to 0^+$.

Hint

Formally

$$\int_{B_{2/(2-\rho)}\setminus B_1} |\hat{x}\cdot\tilde{E}_\rho^+|^p\,dx = \int_{B_{2\rho}\setminus B_\rho} (2-\rho)^p |\hat{y}\cdot E_\rho^+|^p |\det(DF_\rho)|\,dy$$

$\downarrow$ spherical expansion of E_ρ^+

$$\begin{cases} = O(\rho^{-1}) & p=2, \\ \leq O(1) & p=1. \end{cases}$$

suggesting a superposition of Delta functions at the interface!

Distributional limits

Theorem [Lassas-Z]

$$\tilde{E}_\rho \overset{\rho\to 0}{\rightharpoonup} \tilde{E} + \alpha[\tilde{J}]\delta_{\partial B_1}, \quad \tilde{H}_\rho \overset{\rho\to 0}{\rightharpoonup} \tilde{H} + \beta[\tilde{J}]\delta_{\partial B_1}$$

where

$$(\tilde{E}, \tilde{H}) = \begin{cases} (F_*E, F_*H) & 1 < |x| < 2, \\ (E_0, H_0) & |x| < 1 \end{cases}$$

with (E, H) denotes the background waves in the vacuum space and

$$\begin{cases} \nabla \times E_0 = i\omega\mu_0 H_0, \quad \nabla \times H_0 = -i\omega\varepsilon_0 E_0 & \text{on } B_1 \\ \nu \cdot E_0|_{\partial B_1} = \nu \cdot H_0|_{\partial B_1} = 0 \end{cases}$$

- *extraordinary surface voltage effect* [Zhang etc.].

Two dimensional approximate cloaking and non-local (pseudo-differential) boundary conditions

Cloaking for scalar optics and acoustics: Helmholtz equations

- The Helmholtz equation for acoustics or scalar optics, with a source term p, inverse of the anisotropic mass density $\sigma = (\sigma^{jk})$ and the bulk modulus λ

$$\lambda \nabla \cdot \sigma \nabla u + \omega^2 u = p \quad \text{in } \Omega.$$

- **Dirichlet to Neumann map:** $\boxed{\Lambda_{\sigma,\lambda} : u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}}$.
- **Cloaking medium**

$$(\tilde{\sigma}, \tilde{\lambda}) = \begin{cases} (F_* I, F_* 1) & 1 < |x| \le 2 \\ (\sigma_a, \lambda_a) \text{ arbitrary} & |x| \le 1 \end{cases}$$

where $F_* \lambda(x) := [\det(DF)\lambda] \circ F^{-1}(x)$.

- $\tilde{u}$ "=" $u \circ F^{-1}$ in $B_2 \setminus \overline{B_1}$.

Singular ideal cloaking medium in $\mathbb{R}^2$

- Cloaking layer: in $B_2 \backslash \overline{B_1}$

$$\tilde{\sigma} = F_* I = \boxed{\frac{|x| - 1}{|x|}} \Pi(x) + \boxed{\frac{|x|}{|x| - 1}} (I - \Pi(x))$$

$$\tilde{\lambda} = F_* 1 = \frac{|x|}{4(|x| - 1)}$$

- Both degenerate and blow-up singularities at $|x| = 1^+$!

Truncation based regularization scheme

- **Regularized** medium with regularization parameter $1 < R < 2$

$$(\tilde{\sigma}_R, \tilde{\lambda}_R) = \begin{cases} (\tilde{\sigma}, \tilde{\lambda}) & |x| > R \\ (\sigma_a, \lambda_a) & |x| \leq R \end{cases}$$

- We are interested in the limiting behavior of the solution near the interface when *an internal source is present*.

Cloaking a homogeneous medium with an internal source

Suppose (σ_a, λ_a) is constant. Set $\kappa^2 = (\sigma_a \lambda_a)^{-1}$ and $\rho = F^{-1}(R)$

- **Physical space:**

$$(\tilde{\lambda}\nabla \cdot \tilde{\sigma}\nabla + \omega^2)u_R^+ = p, \quad \text{in } B_2 \backslash \overline{B_R}$$

$$\boxed{(\Delta + \kappa^2\omega^2)u_R^- = \kappa^2 p \quad \text{in } B_R}$$

- **Virtual space:** $v_R^+ = u_R^+ \circ F$,

$$\boxed{(\Delta + \omega^2)v_R^+ = p \circ F \quad \text{in } B_2 \backslash \overline{B_\rho}}$$

- Transmission conditions and boundary conditions:

$$\boxed{\begin{array}{l} v_R^+|_{\partial B_\rho^+} = u_R^-|_{\partial B_R^-}, \quad \rho \partial_r v_R^+|_{\partial B_\rho^+} = \kappa R \partial_r u_R^-|_{\partial B_R^-}, \\ v_R^+|_{\partial B_2} = f. \end{array}}$$

Cloaking a homogeneous medium with an internal source

Given $p \in C^\infty(\mathbb{R}^2)$ with $\text{supp}(p) \subset B_{R_0}$ $(0 < R_0 < 1)$ and suppose $f = 0$ on ∂B_2

- Spherical expansions:

$$u_R^-(\tilde{r}, \theta) = \sum_{n=-\infty}^{\infty} (a_n J_{|n|}(\kappa\omega\tilde{r}) + p_n H_{|n|}^{(1)}(\kappa\omega\tilde{r}))e^{in\theta}, \quad \tilde{r} \in (R_0, R)$$

$$v_R^+(r, \theta) = \sum_{n=-\infty}^{\infty} (b_n J_{|n|}(\omega r) + c_n H_{|n|}^{(1)}(\omega r))e^{in\theta}, \quad r \in (\rho, 2)$$

- Linear system about a_n, b_n and c_n by the transmission conditions and boundary condition.

$$a_n = \frac{R(H^{(1)}_{|n|})'(\kappa\omega R)l_1 - \rho H^{(1)}_{|n|}(\kappa\omega R)l_2}{D_n} p_n := \frac{A_n}{D_n} p_n$$

$$b_n = \frac{R\{(H^{(1)}_{|n|})'(\kappa\omega R)J_{|n|}(\kappa\omega R) - J'_{|n|}(\kappa\omega R)H^{(1)}_{|n|}(\kappa\omega R)\}H^{(1)}_{|n|}(3\omega)}{D_n} p_n$$

$$c_n = -\frac{R\{(H^{(1)}_{|n|})'(\kappa\omega R)J_{|n|}(\kappa\omega R) - J'_{|n|}(\kappa\omega R)H^{(1)}_{|n|}(\kappa\omega R)\}J_{|n|}(3\omega)}{D_n} p_n$$

where

$$D_n = \frac{-i2^n\omega^{-n-1}(n-1)!}{\pi} J_n(3\omega)\left[\kappa^2\omega R J'_n(\kappa\omega R) + nJ_n(\kappa\omega R)\right]\rho^{-n} + O(\rho^{-n+1}),$$

Observations for resonant case

Resonant frequency ω for mode n

$\Longleftrightarrow$ cloak-busting inclusion limit $\kappa = (\sigma_0\lambda_0)^{-1/2}$

$\Longleftrightarrow$ $|a_n|, |b_n|, |c_n| \to \infty$ as $R \to 1^+ (\rho \to 0^+)$ $(n \geq 1)$

$$\Longleftrightarrow \boxed{\left[\omega\kappa^2 R (J_{|n|})'(\kappa\omega R) + |n| J_{|n|}(\kappa\omega R)\right]\Big|_{R=1} = 0}$$

$\Longleftrightarrow$ $V_{\pm n}(\tilde{r},\theta) := J_{|n|}(\kappa\omega\tilde{r})e^{\pm in\theta}$ are eigenfunctions of

$$\boxed{\begin{aligned} &(\Delta + \kappa^2\omega^2)V = 0 \quad \text{in } B_1, \\ &[\kappa\tilde{r}\partial_{\tilde{r}}V + (-\partial_\theta^2)^{1/2}V]|_{\tilde{r}=1^+} = 0. \end{aligned}}$$

Non-local boundary conditions

$$[\kappa\tilde{r}\partial_{\tilde{r}}V + (-\partial_\theta^2)^{1/2}V]|_{\tilde{r}=1^+} = 0.$$

- Operator $\mathcal{A} := (-\partial_\theta^2)^{1/2}$ is a pseudo-differential operator: Square root of positive laplacian over $\mathbb{S}^1$.
  Symbol of P:

$$\widehat{Pu} = \mathrm{Sym}(P)\widehat{u}$$

$$\mathrm{Sym}(\nabla) = -i\xi, \quad \mathrm{Sym}(\Delta) = -|\xi|^2, \quad \xi \in \mathbb{R}^n$$

$$\mathrm{Sym}(\mathcal{A}) = |\xi|$$

- A **non-local** boundary condition:

$$\mathcal{A}u = \mathcal{F}^{-1}(|\xi|\widehat{u})$$

Non-resonant result: non-local boundary conditions

Suppose ω and (σ_a, λ_a) satisfy

$$\begin{cases} [\omega\kappa^2 R(J_{|n|})'(\kappa\omega R) + |n| J_{|n|}(\kappa\omega R)]\big|_{R=1} \neq 0, \\ J_{|n|}(2\omega) \neq 0, \end{cases} \qquad \text{for } n \in \mathbb{Z}.$$

Theorem [Lassas-Z]

As $R \to 1^+$, u_R (the solution in the physical space) converges uniformly in compact subsets of $B_2 \backslash \partial B_1$ to the limit u_1 satisfying

$$(\Delta + \kappa^2\omega^2)u_1 = \kappa^2 p \quad \text{in } B_1,$$
$$[\kappa\partial_{\tilde{r}} u_1 + (-\partial_\theta^2)^{1/2} u_1]\big|_{\partial B_1} = 0.$$

- possiblely due to the fact that the phase velocity of the waves in the invisibility cloak approaches infinity near the interface, even though the group velocity stays finite.

Thank you for your attention!