Luminy Lecture 2: Spectral rigidity of the ellipse joint work with Hamid Hezari

Luminy Lecture
April 12, 2015

Isospectral rigidity of the ellipse

The purpose of this lecture is to prove that ellipses are spectrally rigid among C^∞ domains with their $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, i.e. ellipses do not admit isospectral deformations Ω_t through smooth domains with two symmetries.

Good feature: the competing domains are smooth, not just real analytic;

Bad feature: the symmetry (is it necessary or just a technical artifice?)

This is a step in the direction of the

Conjecture Ellipses are spectrally determined: one can hear the shape of an ellipse .

Elliptical billiards

Elliptical billards are very special and it is conjectured (originally by Birkhoff) that they are the unique billiard tables for which the billards are *completely integrable*.

One can find movies of elliptical billiards on the website http://cage.ugent.be/ hs/billiards/billiards.html

We start with background on billiards.

Billard flow and billiard map on domains with boundary

To define the geodesic or billiard flow G^t on a domain Ω with boundary $\partial\Omega$, we need to specify what happens when a geodesic intersects the boundary.

We denote by $S^*\Omega$ the unit tangent vectors to the interior of Ω and by $S^*_{in}\partial\Omega$ the manifold with boundary of inward pointing unit tangent vectors to Ω with footpoints in $\partial\Omega$. The boundary consists of unit vectors tangent to $\partial\Omega$. The billiard flow G^t is a flow on $S^*\Omega \cup S^*_{in}\partial\Omega$, defined as follows: When an interior geodesic of Ω intersects the boundary $\partial\Omega$ transversally, it is reflected by the usual Snell law of equal angles. Such trajectories are called (transveral) reflecting rays. The complications occur when a geodesic intersects the boundary tangentially in $S^*\partial\Omega$.

Convex versus non-convex domains, creeping rays

Convex domains are simpler than non-convex domains, since interior rays cannot intersect the boundary tangentially. It should be noted that geodesics of $\partial\Omega$ with the induced metric are important billiard trajectories. They are limits of 'creeping rays', i.e. rays with many small links (interior segments) which stay close to $\partial\Omega$. In particular, the boundary of a convex plane domain is a closed billiard trajectory.

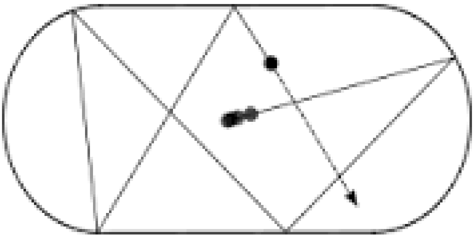
Billiard map or boundary map

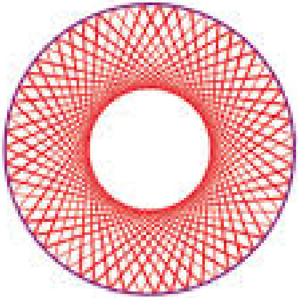
The set $S^*_{in}\partial\Omega$ of inward pointing unit vectors behaves like a global cross section to the billiard flow. It is then natural to reduce the dimension by defining the *billiard ball map* $\beta : B^*(\partial\Omega) \to B^*(\partial\Omega)$, where $B^*(\partial\Omega)$ is the ball bundle of the boundary. We first identify $S^*_{in}\partial\Omega \simeq B^*(\partial\Omega)$ by adding to a tangent (co)vector $\eta \in B^*_q\partial\Omega$ of length < 1 a multiple $c\nu_q$ of the inward point unit normal ν_q to form a covector in $S^{in}_{\partial\Omega}\Omega$. The image $\beta(q, v)$ is then defined to be the tangential part of the first intersection of $G^t(q, \eta + c\nu_q)$ with $\partial\Omega$. The billiard map is symplectic with respect to the natural symplectic form on $B^*(\partial\Omega)$.

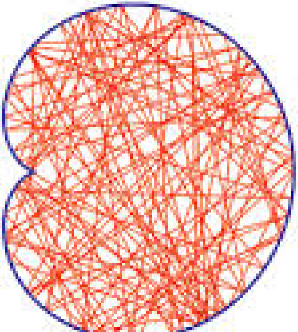
Billiard map of a plane domain

An equivalent description of the billiard map of a plane domain is as follows. Let $q \in \partial\Omega$ and let $\varphi \in (0, \pi)$. The point (q, φ) corresponds to an inward pointing unit vector making an angle φ with the tangent line, with $\varphi = 0$ corresponding to a fixed orientation (say counter-clockwise). The billiard map is then $\beta(q, \varphi) = (q', \varphi')$ where $(q`, \varphi')$ are the parameters of the reflected ray at the first point of intersection with the boundary. The map β is then area preserving with respect to $\sin \varphi ds \wedge d\varphi$.

Billiards in a stadium, cardiod and annulus







Length spectrum of a manifold with boundary

In the case of Euclidean domains, or more generally domains where there is a unique geodesic between each pair of boundary points, one can specify a billiard trajectory by its successive points of contact $q_0, q_1, q_2, \ldots$ with the boundary. The *n-link periodic reflecting rays* are the trajectories where $q_n = q_0$ for some $n > 1$. The point $q_0, \ldots, q_n$ is then a critical point of the length functional

$$\mathcal{L}(q_0, \ldots, q_n) = \sum_{i=0}^{n-1} |q_{i+1} - q_i|$$

on $(\partial\Omega)^n$.

Length spectrum of a manifold with boundary

In the boundary case, the length spectrum $Lsp(\Omega)$ is the set of lengths of closed billiard trajectories; it is not discrete, but rather has points of accumulation at lengths of trajectories which have intervals along the boundary. In the case of convex plane domains, e.g., the length spectrum is the union of the lengths of periodic reflecting rays and multiples of $|\partial\Omega|$. According to the standard terminology, $Lsp(M, g)$ is the set of distinct lengths, not including multiplicities, and one refers to the the length spectrum repeated according to multiplicity as the *extended length spectrum*.

Elliptical billiards are completely integrable

Let $E_{a,b}$ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Its foci are at $\pm\sqrt{a^2 - b^2}$ where $a > b$.

Let $0 < Z \leq b$, and define the confocal ellipse

$$E_Z = \frac{x^2}{\epsilon + Z} + \frac{y^2}{Z} = 1.$$

PROPOSITION

Let $p \in E_{a,b}$ and let ℓ, ℓ' be two lines from p which are tangent to E_Z. The ℓ, ℓ' make equal angles with ν_p, the normal at p. Similarly for the confocal hyperbolae with $-\epsilon < Z < 0$.

Orbits through a focus

There is a bouncing ball orbit on the major axis through the two foci.

Now consider another billiard trajectory starting at some point $p \in \partial E_{a,b}$ and going through a focus, say F_2.

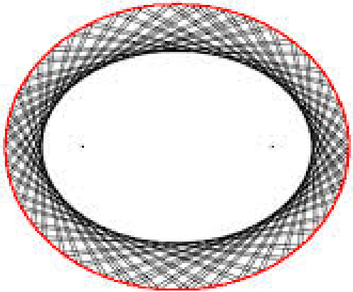
Lemma

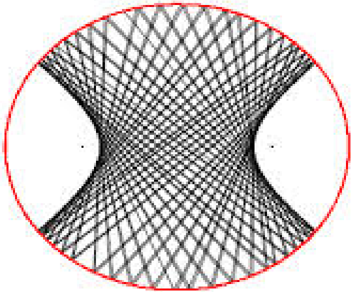
An orbit that goes through one focus will go infinitely often through both foci and will asymptotically tend to the bouncing ball orbit through the two foci.

Elliptical billiards are completely integrable

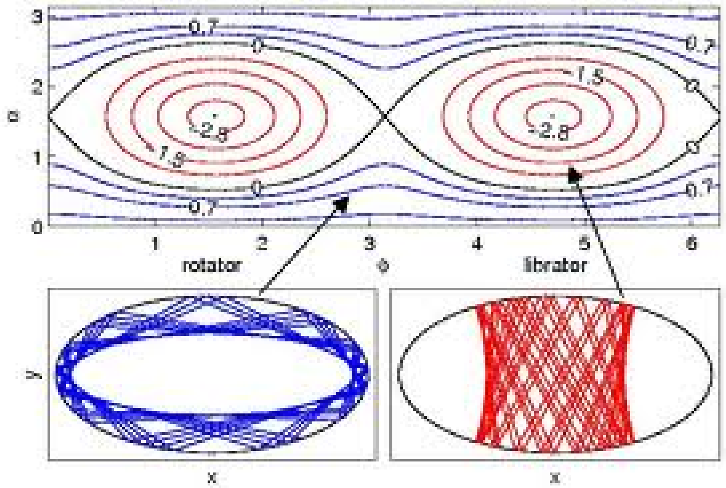
Both the billiard flow and billiard map of the ellipse are completely integrable. I.e. there exist foliations of $S^*\Omega$ by invariant tori for Φ^t and of $B^*\partial\Omega$ for β.

Caustics of elliptical billiards





Phase portrait of billiard map

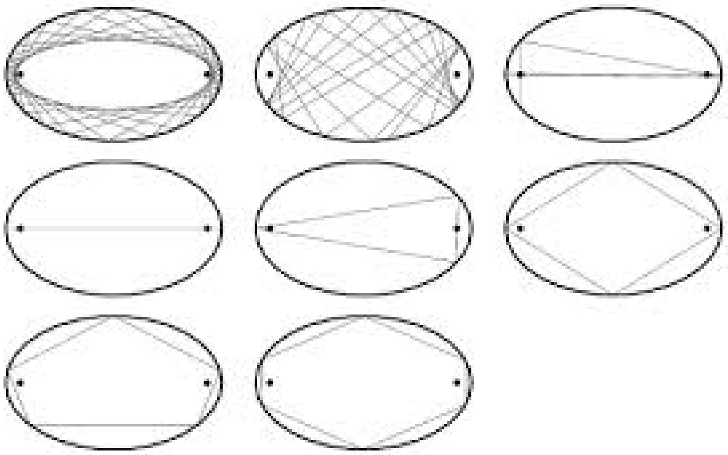


Invariant curves of periodic orbits

Except for certain exceptional trajectories, the periodic points of period T form Lagrangian tori in $S^*\Omega$, which intersect $B^*\partial\Omega$ in invariant curves for β.

The exceptions are the two bouncing ball orbits through the major/minor axes and the trajectories which intersect the foci or glide along the boundary.

Periodic orbits of elliptical billiards come in one-parameter families



Spectral rigidity of an ellipse

Our main result is the infinitesimal spectral rigidity of ellipses among C^∞ plane domains with the symmetries of an ellipse. We orient the domains so that the symmetry axes are the x-y axes. The symmetry assumption is then that ρ_s is invariant under $(x, y) \to (\pm x, \pm y)$.

Before stating the result, we review the definition.

Isospectral deformation

An isospectral deformation of a plane domain Ω_0 is a one-parameter family Ω_s of plane domains s.th. the $Spec_B \Delta_s$ is constant (including multiplicities) for a fixed boundary condition B.

We assume that $\Omega_s = \varphi_s(\Omega_0)$ where φ_s is a one-parameter family of diffeomorphisms of a ball containing Ω_0. Also let $X = \frac{d\varphi_s}{ds}$. The normal component of X on $\partial\Omega_0$ is denoted X_N.

Deformations

For expository clarity, we think of X as a normal vector field $\dot{\rho}(q)\nu_q$ on $\partial\Omega$. We then think of $\partial\Omega_t$ as the image under the map

$$x \in \partial\Omega_0 \to x + \rho_s(x)\nu_x, \tag{1}$$

where $\rho_s \in C^1([0, s_0], C^\infty(\partial\Omega))$. The first variation is defined to be $\dot{\rho}(x) = \delta\rho\,(x) := \frac{d}{ds}|_{s=0}\rho_s(x)$.

Infinitesimal rigidity

An isospectral deformation is said to be trivial if $\Omega_s = \Omega_0$ (up to isometry) for sufficiently small s. A domain Ω_0 is said to be spectrally rigid if all isospectral deformations are trivial.

Even if the domains Ω_s or the $\rho_s(x)$ are C^∞ for each s, we need to consider the dependence of $\rho_s(x)$ in s.

A deformation is said to be a C^1 deformation through C^∞ domains if each Ω_s is a C^∞ domain and the map $s \to \Omega_s$ is C^1.

Spectral rigidity of an ellipse

THEOREM

Suppose that Ω_0 *is an ellipse, and that* Ω_s *is a* C^1 *Dirichlet (or Neumann) isospectral deformation of* Ω_0 *through* C^∞ *domains with* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *symmetry. Let* ρ_s *be as in (1). Then* $\dot{\rho} = 0$.

Consequently, there exist no non-trivial real analytic curves Ω_t *of* C^∞ *of domains with the spectrum of an ellipse.*

Infinitesimal rigidity versus rigidity

Indeed, all isospectral deformations would have to be "flat" at $\epsilon = 0$.

COROLLARY

Suppose that Ω_0 is an ellipse, and that $s \to \Omega_s$ is a C^∞ Dirichlet (or Neumann) isospectral deformation through $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric C^∞ domains. Then ρ_s must be flat at $s = 0$.

C^∞ and not analytic

The main advance is that the domains Ω_s are allowed to be C^∞ rather than real analytic. Much less than C^∞ could be assumed for the domains Ω_s, e.g. C^6 might be enough.

It would be desirable to remove the symmetry assumption (to the extent possible), but symmetry seems quite necessary for the argument.

Much of the argument is completely general– and applies to any convex plane domain. Only the very last step involves ellipses.

Outline of proof

1. Hadamard variational formula for the Diriclet (resp. Neumann) wave kernel $\cos t\sqrt{\Delta_B})$ under variation of the boundary, and in particular for its (regularized) trace. This is completely general–any domain, any manifold, any dimension.

2. Proof that $\delta Tr \cos t\sqrt{\Delta_B})$ has co-normal singularities at lengths of periodic transversal reflecting rays. Again, completely general.

3. Symbol at special lengths is an Abel transform.
Guillemin-Melrose: vanishing of Abel transform at special lengths implies rigidity. Only this step uses the ellipse.

Notation

Below, we denote the perimeter of Ω by $|\partial\Omega|$. We also denote by $Lsp(\Omega)$ the length spectrum of Ω, i.e. the set of lengths of closed billiard trajectories.
By $Lsp(\Omega_0)$ we mean the length spectrum of the ellipse, i.e. the set of lengths of periodic billiard trajectories. They come in one dimensional families, which intersect $B^*\partial\Omega_0$ in invariant curves Γ. There is a natural Leray measure on each invariant curve of periodic orbits which we denote by $d\mu_\Gamma$.

Billiards

We denote by $\Phi^t : S^*\Omega \to S^*\Omega$ the generalized geodesic flow (or broken billiard flow) of the ellipse Ω_0, and we denote by $\beta : B^*\partial\Omega_0 \to B^*\partial\Omega_0$ the associated billiard map. The broken geodesic flow extends by homogneneity (degree one) to $T^*\Omega - 0$. We denote the Hamiltonian vector field of the Euclidean norm function g by H_g.

Wave trace of an ellipse

if Ω is isospectral to an ellipse $\mathcal{E}_e$, then the wave trace singularities at lengths of closed billiard trajectories must be the same as for the ellipse. The wave trace for the ellipse has the form,

$$Tr \cos t\sqrt{\Delta_g} = e_0(t) + \sum_{\mathcal{T}} e_{\mathcal{T}}(t) \tag{2}$$

where $e_0(t) = C_2 \; Vol(M,g) \; (t + i0)^{-2} + \dots$ at $t = 0$, where $\{\mathcal{T}\}$ runs over the connected components of the set of periodic billiard trajectories,where $L_{\mathcal{T}}$ is the length of the periodic trajectories in the component $\mathcal{T}$, and where

$$e_{\mathcal{T}} = c_{\mathcal{T},\frac{3}{2}}(t - L_{\mathcal{T}} + i0)^{-\frac{3}{2}} + c_{\mathcal{T},\frac{1}{2}}(t - L_{\mathcal{T}} + i0)^{-\frac{1}{2}} + \dots . \tag{3}$$

In the non-degenerate case, the leading exponent would be -1, not $-3/2$.

Proof: Hadamard variational formula for trace of wave group

Proposition

For each $T \in Lsp(\Omega_0)$ for which all billiard trajectories are transverse reflecting rays, there exist constants C_Γ independent of $\dot{\rho}$ such that, near T, the leading order singularity is

$$\delta \; Tr \; e^{it\sqrt{\Delta}}$$

$$\sim it \sum_{\Gamma : L_\Gamma = T} C_\Gamma \int_\Gamma \dot{\rho} \, \gamma \, d\mu_\Gamma \, (t - T + i0)^{-\frac{5}{2}},$$

where the sum is over the sets Γ of points on periodic trajectories of period T; γ is a certain function.

Level sets of action

The fixed point set of a given period T is a certain level set $\{I = \alpha_T\}$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant Hamiltonian on $B^*\partial\Omega_0$,

$$I := p_\vartheta^2 + c^2 \cos^2 \vartheta.$$

The level sets $\{I = \alpha\}$ are β-invariant curves and up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry they are irreducible invariant curves, i.e. are not unions of invariant components.

Leray form

There is a natural invariant measure $d\mu_\alpha$ on each component of $\{I = \alpha\}$, namely the Leray quotient measure $d\mu_\alpha = \frac{d\vartheta \wedge dp_\vartheta}{dI}$ of the symplectic area form by dI. They are invariant under the Hamilton flow of I and under the billiard map β.
In the case of an ellipse, the fixed point sets are clean fixed point sets for Φ^t in $T^*\Omega$, resp. for β in $B^*\partial\Omega$ (Guillemin-Melrose).

Ideas of proof, II: principal symbol of HD variation of wave trce

Lemma

Let Ω_0 be an ellipse, and $T \in Lsp(\Omega_0)$ with T not a multiple of $|\partial\Omega|$. Then the principal symbol of $Tr \dot{\rho}(\Delta^{-\frac{1}{2}} U)^b(t)$ at $t = T$ is given by $\int_{I=\alpha} \dot{\rho}\, \gamma\, d\mu_\alpha$, in the Dirichlet case, where $d\mu_\alpha$ is the Leray measure on $\{I = \alpha\}$.

A kind of length spectral simplicity

Proposition

(Guillemin-Melrose): Let $T_0 = |\partial\Omega_0|$. *Then for every interval* $(mT_0 - \epsilon, mT_0)$ *for* $m = 1, 2, 3, \ldots$ *there exist infinitely many periods* $T \in Lsp(\Omega_0)$ *for which* Γ_T *is the union of two invariant curves which are mapped to each other by* $\theta \to \pi - \theta$.

Corollary for wave trace coefficients

Since we assume $\dot{\rho}$ to have the same symmetry, we obtain:

COROLLARY

If $\dot{\varphi}$ is the velocity of an isospectral deformation, then

$$\int_{\Gamma_T} \dot{\rho}\, \gamma \, d\mu_T = 0$$

for each T for which Γ_T is the union of two invariant curves which are mapped to each other by $\theta \to \pi - \theta$.

Proof of Theorem

The remainder of the proof is the same as one of Guillemin-Melrose.

Proposition

The only $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant function $\dot{\rho}$ satisfying the equations of Corollary 8 is $\dot{\rho} = 0$.

First, we may assume $\dot{\rho} = 0$ at the endpoints of the major/minor axes, since the deformation preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and we may assume that the deformed bouncing ball orbits are aligned with the original ones. Thus $\dot{\rho}(\pm\sqrt{a}) = \dot{\rho}(\pm\sqrt{b}) = 0$.

Injectivity of an Abel transform

The Leray measure may be explicitly evaluated. By a change of variables with Jacobian J, the integrals become

$$F(Z) = \int_a^b \frac{\dot{\rho}(t)\,\gamma\,J(t)dt}{\sqrt{t-(b-Z)}}. \tag{4}$$

for an infinite sequence of Z accumulating at b. Since $0 < a < b$, the function $F(Z)$ is smooth in Z for Z near b.

Injectivity of an Abel transform

It vanishes infinitely often in each interval $(b - \epsilon, b)$, hence is flat at b. The kth Taylor coefficient at b is

$$F^{(k)}(b) = \int_a^b \dot{\rho}(t)\, \gamma\, J(t) t^{-k-\frac{1}{2}} dt = 0. \tag{5}$$

Since the functions t^{-k} span a dense subset of $C[a, b]$), it follows that $\dot{\rho} \equiv 0$.

Details of HD variation of wave trace

THEOREM

Let $\Omega_0 \subset \mathbb{R}^n$ be a C^∞ Euclidean domain with the property that the fixed point sets of the billiard map are clean. Then, for any C^1 variation of Ω_0 through C^∞ domains Ω_ϵ, the variation of the wave traces $\delta Tr e^{it\sqrt{-\Delta_\epsilon}}$, with Dirichlet (or Neumann) boundary conditions is a classical co-normal distribution for $t \neq m|\partial\Omega_0|$ ($m \in \mathbb{Z}$) with singularities contained in $Lsp(\Omega_0)$. For each $T \in Lsp(\Omega_0)$ for which the set Γ_T of periodic points of the billiard map β of length T is a d-dimensional clean fixed point set consisting of transverse reflecting rays, there exist non-zero constants C_Γ independent of $\dot\rho$ such that, near T, the leading order singularity is

$$\delta \; Tr \; e^{it\sqrt{-\Delta_\epsilon}} \sim \left(it \sum_{\Gamma : L_\Gamma = T} C_\Gamma \int_\Gamma \dot\rho \, \gamma \; d\mu_\Gamma \right) (t - T + i0)^{-2-\frac{d}{2}},$$

modulo lower order singularities.

Applications to deformations of the ellipse

For any C^1 variation of an ellipse through C^∞ domains Ω_ϵ, the leading order singularity of the wave trace variation is,

$$\delta \; Tr \; e^{it\sqrt{-\Delta_\epsilon}} \sim \left(it \sum_{\Gamma : L_\Gamma = T} C_\Gamma \int_\Gamma \dot{\rho} \, \gamma \, d\mu_\Gamma \right) (t - T + i0)^{-\frac{5}{2}},$$

modulo lower order singularities, where the sum is over the components Γ of the set Γ_T of periodic points of β of length T.

Hadamard variational formula for wave traces

Consider the Dirichlet (resp. Neumann) eigenvalue problems for a one parameter family of smooth Euclidean domains $\Omega_\epsilon \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta_{B\epsilon}\Psi_j(\epsilon) = \lambda_j^2(\epsilon)\Psi_j(\epsilon) \text{ in } \Omega_\epsilon, \\ \\ B\Psi_j(\epsilon) = 0, \end{cases} \tag{6}$$

where the boundary condition B could be $B\Psi_j(\epsilon) = \Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Dirichlet) or $\partial_{\nu_\epsilon}\Psi_j(\epsilon)|_{\partial\Omega_\epsilon}$ (Neumann). Here, $\lambda_j(\epsilon)$ are the eigenvalues of Δ_ϵ, enumerated in order and with multiplicity, and ∂_{ν_ϵ} is the interior unit normal to Ω_ϵ. We do not assume that $\Psi_j(\epsilon)$ are smooth in ϵ. We now review the Hadamard variational formula for the variation of Green's kernels, and adapt the formula to give the variation of the (regularized) trace of the wave kernel.

Notation

We denote by $U_B(t) = e^{it\sqrt{-\Delta_{B\epsilon}}}$ the wave group of Ω_ϵ with boundary conditions B. We could as easily (or more easily) work with

$$E_B(t) = \cos\left(t\sqrt{-\Delta_{B\epsilon}}\right), \quad S_B(t) = \frac{\sin\left(t\sqrt{-\Delta_{B\epsilon}}\right)}{\sqrt{-\Delta_{B\epsilon}}}. \qquad (7)$$

Since the boundary conditions are fixed in the deformation, we often omit the subscript for them, and only include it when the formulae depend on the choice. We recall that $U_B(t)$ has a distribution trace as a tempered distribution on $\mathbb{R}$. That is, $U_B(\hat{\rho}) = \int_{\mathbb{R}} \hat{\rho}(t) U_B(t) dt$ is of trace class for any $\hat{\rho} \in C_0^\infty(\mathbb{R})$.

Notation

We further denote by dS the surface measure on the boundary $\partial\Omega$ of a domain Ω, and by $ru = u|_{\partial\Omega}$ the trace operator. We further denote by $r^D u = \partial_\nu u|_{\partial\Omega}$ the analogous Cauchy data trace for the Dirichlet problem. We simplify the notation for the following boundary traces $K^b(q, q') \in \mathcal{D}'(\partial\Omega \times \partial\Omega)$ of a Schwartz kernel $K(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ (or more precisely a distribution defined in a neighborhood of $\partial\Omega \times \partial\Omega$): $K^b(q, q')$ is given for D reps. N BC by

$$\begin{cases} (r_q^D r_{q'}^D K)(q, q') = \big(r_q r_{q'} N_{\nu_q} N_{\nu_{q'}} K\big)(q, q'), \\ \\ (r_q^N r_{q'}^N K)(q, q') = (\nabla_q^T \nabla_{q'}^T r_q r_{q'} K)(q, q') - \big(r_q r_{q'} \Delta_x K\big)(q, q'), \end{cases}$$

Here, the subscripts q, q' refer to the variable involved in the differentiating or restricting. Also, N_ν is any smooth vector field in Ω extending ν.

Notation

We are principally interested in $K(x,y) = (-\Delta_{x,B})^{-\frac{1}{2}} U_B(t,x,y)$. In the Dirichlet, resp. Neumann, case then we have,

$$((-\Delta_{x,B})^{-\frac{1}{2}} U_B)^b(t,q,q')$$

$$= r_q^D r_{q'}^D (-\Delta_{x,D})^{-\frac{1}{2}} U_D(t,q,q'), \text{ resp.}$$

$$\nabla_q^T \nabla_{q'}^T r_q r_{q'} (-\Delta_{x,N})^{-\frac{1}{2}} U_N(t,q,q') - r_q r_{q'} ((-\Delta_{x,N})^{\frac{1}{2}} U_N)(t,q,q').$$

HD variation of the wave trace

Lemma

The variation of the wave trace with boundary conditions B is given by,

$$\delta\ Tr\ U_B(t) = \frac{it}{2}\int_{\partial\Omega_0}((-\Delta_B)^{-\frac{1}{2}}U_B)^b(t,q,q)\dot{\rho}(q)dq.$$

In particular,

$$\delta\ Tr\ E_B(t) = -\frac{t}{2}\int_{\partial\Omega_0} S_B^b(t,q,q)\dot{\rho}(q)dq.$$

We summarize by writing,

$$\delta\ Tr\ U_B(t) = \frac{it}{2} Tr_{\partial\Omega_0}\dot{\rho}((-\Delta_B)^{-\frac{1}{2}}U_B)^b.$$

Classical HD variational formulae

In the Dirichlet case, the classical Hadamard variational formulae states that, under a sufficiently smooth deformation Ω_ϵ,

$$\delta G_D(\lambda, x, y) = -\int_{\partial\Omega_0} \frac{\partial}{\partial\nu_2} G_D(\lambda, x, q) \frac{\partial}{\partial\nu_1} G_D(\lambda, q, y) \dot{\rho}(q)\, dq. \tag{8}$$

Proof cont.

We derive the Hadamard variational formulae for wave traces from that of the Green's function by using the identities,

$$\lambda R_B(\lambda) = \int_0^\infty e^{-i\lambda t} E_B(t) dt, \quad \frac{d}{dt} S_B(t) = E_B(t) \tag{9}$$

integrating by parts and using the finite propagation speed of $S_B(t)$ to eliminate the boundary contributions at $t = 0, \infty$. It follows that

$$R_B(\lambda) = i \int_0^\infty e^{-i\lambda t} \; S_B(t) dt. \tag{10}$$

Singularities of Hadamard variation of trace

We now study the singularity expansion of $\delta Tr e^{it\sqrt{-\Delta_\epsilon}}$.

In the Dirichlet case,

$$Tr_{\partial\Omega}\dot{\rho}((-\Delta_D)^{-\frac{1}{2}}U_D)^b = \pi_* \; \dot{\rho} \; \Delta^*\big(r_1 r_2 N_{\nu_1} N_{\nu_2}(-\Delta)^{-\frac{1}{2}} U_D(t,x,y)\big), \tag{11}$$

where N_{ν_1} is any smooth vector field in Ω extending ν. Here, $r_1 u(\cdot, x_2) = u(q, x_2)(q \in \partial\Omega)$ is the restriction of u in the first variable to the boundary; similarly for r_2. Also, $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ is the diagonal embedding $q \to (q,q)$ and π_* (the pushforward of the natural projection $\pi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$) is the integration over the fibers with respect to arc-length dq. Since $(-\Delta)^{-\frac{1}{2}}U(t,x,y)$ is microlocally a Fourier integral operator near the transversal periodic reflecting rays of Γ_T, it follows from (11) that the trace is locally a Fourier integral distribution near $t = L$.