Luminy Lecture 1: The inverse spectral problem

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The inverse spectral problem

The goal of the lectures is to introduce the ISP = the inverse spectral problem. An inverse spectral problem is to invert the map

$$Sp : \text{Some operators with discrete spectrum} \to \text{their spectra.}$$

The operators are parametrized by domains, metrics, or potentials and the ISP is to determine the domain, metric or potential. E.g.:

1. Laplacians on bounded smooth domains $\Omega \subset \mathbb{R}^n$ with Dirichlet or Neumann boundary conditions: Determine Ω from $Sp(\Delta_\Omega)$;
2. Metric Laplacians Δ_g on compact Riemannian manifolds (M, g): Determine g from $Sp(\Delta_g)$;
3. Schrödinger operators $-h^2\Delta + V$ (Laplacian plus a potential): Determine V from $Sp(-h^2\Delta + V)$;
4. Scattering matrices $S_V(h) : L^2(S^n) \to L^2(S^n)$.

Can you hear the shape of a drum

The most famous ISP is the Kac problem: 'Can you hear the shape of a drum'? I.e. can you determine a bounded plane domain $\Omega \subset \mathbb{R}^2$ from the spectrum $\{\lambda_j\}$ of the boundary value problem:

$$\begin{cases} \Delta\varphi_j = \lambda_j^2 \varphi_j \text{ in } \Omega, \\ \\ B\varphi_j|_{\partial\Omega} = 0, \end{cases}$$

Here, Δ is the Euclidean Laplacian. The boundary condition is Dirichlet $B\varphi = \varphi$, or Neumann $B\varphi = \partial_\nu \varphi$ (the normal derivative).

Can one determine Ω from $Sp(\Delta_B) = \{\lambda_j^2\}$?

Outline of lectures on the ISP (inverse spectral problem)

1. Rapid review of current status of the ISP.
2. Define local spectral invariants: Heat and wave invariants.
3. Calculation of wave invariants and the Hadamard parametrix for the wave group on manifolds without conjugate points.
4. Bounded Euclidean domains: wave equation, wave invariants and Melrose-Marvizi invariants.
5. Rigidity of ellipses. Recent dynamical result of Avila-de Simoi - Kaloshin and a new application to the ISP.

Surveys used in these lectures

- K. Datchev and H. Hezari, Inverse problems in spectral geometry, *Inside Out II*, p. 455-485, Math. Sci. Res. Inst. Publ., 60, Cambridge Univ. Press, Cambridge, 2013.
- S.Z. Survey on the inverse spectral problem, Notices of the ICCM, Vol. 2 (2) (2014) 1-20.
- S. Z, The inverse spectral problem. With an appendix by Johannes Sjöstrand and Maciej Zworski. Surv. Differ. Geom., IX, Surveys in differential geometry. Vol. IX, 401467, Int. Press, Somerville, MA, 2004.

Some articles used in these lectures

- C. Gordon, D. Webb and S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, Invent. Math. 110 (1992) 1-22.
- (H. Hezari and S.Z.), C^∞ spectral rigidity of the ellipse, Analysis & PDE 5-5 (2012), 1105–1132 . (arXiv:1007.1741).
- A. Avila, J. de Simoi and V. Kaloshin, An integrable deformation of an ellipse of small eccentricity is an ellipse, arXiv 1412.2853v1.
- R. B. Melrose and S. Marvizi, Spectral invariants of convex planar regions. J. Differential Geom. 17 (1982), no. 3, 475-502.
- G. Popov, Invariants of the length spectrum and spectral invariants of planar convex domains. Comm. Math. Phys. 161 (1994), no. 2, 335-364.
- S. Zelditch, "Inverse spectral problem for analytic domains, II: Z2-symmetric domains", Ann. of Math. (2) 170:1 (2009), 205-269.

Spectrum Map

The spectrum of a compact Riemannian manifold (possibly with boundary) defines a map

$$Spec : \mathcal{M} \to \mathbb{R}_+^{\mathbb{N}}, \quad (g, B) \to Spec(\Delta_{g,B}) = \{\lambda_0 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots\}$$

from some class of metrics $\mathcal{M}$ on a manifold M to the spectrum of its Laplacian,

$$\begin{cases} \Delta\varphi_j = \lambda_j^2 \varphi_j, \ \ \langle \varphi_i, \varphi_j \rangle = \delta_{ij} \\ \\ B\varphi_j = 0 \ \text{ on } \partial M, \end{cases}$$

with boundary conditions $B : C^\infty(M) \to C^\infty(\partial M)$ if $\partial M \neq \emptyset$. Eigenvalues are repeated according to their multiplicities.

Laplacian

Here, Δ denotes the *positive* Laplacian

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} g^{ij} g \frac{\partial}{\partial x_j}$$

of a Riemannian manifold (M, g), where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$ and $g = \det[g_{ij}]$. We will only consider Dirichlet $Bu = u|_{\partial M}$ and Neumann $Bu = \partial_\nu u|_{\partial M}$.

Isospectrality

Two metrics or domains are called isospectral if their Laplacians have the same spectrum.

Isometric metrics or domains obviously have the same spectrum, and this kind of isospectrality is 'trivial'.

The purpose of inverse spectral theory is to determine as much as possible of (M, g) from its spectrum.

Positive results and counterexamples

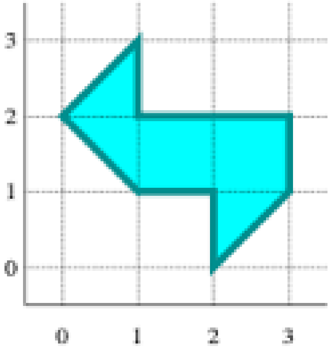
From $Sp(\Delta_g)$ one can recover $Vol(M, g)$, for instance. The volume is called a "spectral invariant". Showing that some geometric invariant is a spectral invariant is called "positive result".

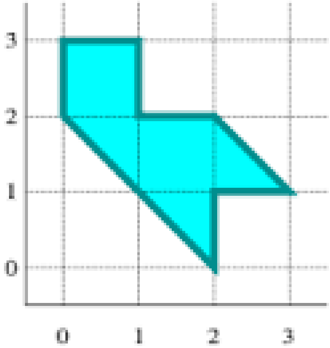
A negative result is one showing that some geometric or topological invariant is not a spectral invariant.

A dramatic kind of negative result is exhibiting non-trivial pairs (or families) of isospectral (M, g).

The question, 'can you hear the shape of a drum', depends on what we mean by 'a drum'. I.e. what is the competing class of domains. There exist counterexamples of Gordon-Webb-Wolpert but they have corners and are non-convex.

Gordon-Webb-Wopert examples





Gordon-Webb-Wopert examples

Restricted isospectral problems

The spectral map

$$Spec : \mathcal{M} \to \mathbb{R}_+^{\mathbb{N}}, \quad (g, B) \to Spec(\Delta_{g,B}) = \{\lambda_0 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots\}$$

is often restricted to some sub-class of metrics or domains. For instance, one may restrict to real analytic domains, or domains with a symmetry. The problem is to show that *Spec* is $1 - 1$ on the sub-class.

It is known that *Spec* is 1-1 on real analytic plane domains with a symmetry among other domains in this class.

It is plausible that one can hear the shape of an of a convex analytic 2 D drum. Here, the class $\mathcal{M}$ is that of convex analytic plane domains.

Positive result with one up/down symmetry

Define the class $\mathcal{D}_{1,L}$ of bounded simply connected C^ω plane domains Ω satisfying:

- Ω is invariant under $(x, y) \mapsto (x, -y)$ which reverses a bouncing ball orbit γ of length $L_\gamma = 2L$.
- The lengths L_γ and $2L_\gamma$ are of multiplicity one in the length spectrum $Lsp(\Omega)$.
- If $\{e^{\pm i\alpha}\}$ are the eigenvalues of the linear Poincaré map P_γ, we also require that $\cos(\alpha/2) \notin \{0, 1, \pm 2\}$. This automatically implies the non-degeneracy of the orbits γ and γ^2.

THEOREM

(S. Z. 2009) The map from $\Omega \in \mathcal{D}_{1,L}$ to its Dirichlet spectrum is 1-1. Same for Neumann.

Isospectral deformations and spectral rigidity

An isospectral deformation of a Riemannian manifold (possibly with boundary) is one-parameter family of metrics satisfying $Spec(M, g_t) = Spec(M, g_0)$ for each t. Similarly, an isospectral deformation of a domain with a fixed background metric g_0 and boundary conditions B is a family Ω_t with $Spec_B(\Omega_t) = Spec_B(\Omega)$. One could also vary boundary conditions.

A metric or domain is called "spectrally rigid" if it does not admit a non-trivial isospectral deformation.

Theorem

(S.Z., in part with H. Hezari)

- *Ellipses are spectrally rigid among C^∞ domains with the same $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry as the ellipse.*
- *Ellipses of small eccentricity are spectrally determined among real analytic domains C^1 close to ellipses.*

Terminology

The basic distinctions in inverse spectral theory are the following. We say that

- a metric or domain is *spectrally determined* (within $\mathcal{M}$) if it is the unique element of $\mathcal{M}$ with its spectrum;
- it is *locally spectrally determined* if there exists a neighborhood of the metric or domain in $\mathcal{M}$ on which it is spectrally determined;
- a metric or domain is *spectrally rigid* in $\mathcal{M}$ if it does not admit an isospectral deformation within the class;
- the inverse spectral problem is solvable in $\mathcal{M}$ if $Spec|_{\mathcal{M}}$ is $1-1$, i.e. if any other metric or domain in $\mathcal{M}$ with the same spectrum is isometric to it. If not, one has found a counterexample.

Easy positive results

It is currently known that

- The genus of a surface is a spectral invariant.
- The round metric on S^2 is spectrally determined. Same for S^n for $n \geq 6$.
- Flat 2-tori or constant curvature -1 hyperbolic surfaces of genus $g \geq 2$ are spectrally rigid. They are not always spectrally determined.
- Discs are spectrally determined among smooth plane domains.

Open problems

It is currently not known if:

- if the standard metric g_0 on S^n is determined by its spectrum (in dimensions $n \geq 7$), i.e. if (M, g) (or even (S^n, g)) is isospectral to (S^n, g_0) then it is isometric to it. This has been proved in dimensions ≤ 6.
- if hyperbolic manifolds are determined by their spectra in dimensions ≥ 3. I.e. if (M_0, g_0) is hyperbolic and (M, g) is isospectral to it, then is (M, g) hyperbolic?
- if flat metrics are determined by their spectra in the sense that if (M, g_0) is flat and (M, g) is isospectral to it, then (M, g) is flat (it is known that this is true in dimensions ≤ 6 or in all dimensions if additionally g is assumed to lie in a sufficiently small neighborhood of g_0.

Strategies for solving the inverse spectral problem

The strategy for obtaining positive results is to:

(A) Define a lot of spectral invariants;
(B) Calculate them in terms of geometric or dynamical invariants;
(C) Try to determine the metric or domain from the geometric invariants.

Recall that $Sp(\Delta) = \{\lambda_j^2\}$. A complete set of spectral invariants is given by traces

$$\begin{cases} \text{The heat trace,} & Z(t) = Tre^{-t\Delta} = \sum_{j=0}^{\infty} e^{-\lambda_j^2 t} \ \ (t > 0), \\ \text{The zeta function} & \zeta(s) = Tr\Delta^{-s} = \sum_{j=0}^{\infty} \lambda_j^{-2s} \ \ (\Re s > n) \\ \text{The wave trace} & S(t) = Tre^{it\sqrt{\Delta}} = \sum_{j=0}^{\infty} e^{i\lambda_j t}, \text{ or} \\ & S_{ev}(t) = Tr\cos t\sqrt{\Delta}. \end{cases} \tag{1}$$

Spectral invariants versus geometry

A key problem is to relate these traces to geometric invariants.
invariants.

Some important spectral invariants such as λ_1 and $\log\det\Delta = -\zeta'(0)$ are hard to relate to geometry.

Local (or residual) spectral invariants

The computable spectral invariants arise from the *singularities* of the traces defined above or, in another language, from non-commutative residues of $\cos t\sqrt{\Delta}$. These are always local invariants. All local invariants known to the author are *wave invariants*, i.e. arising from the singularities of the distribution trace or residues of the wave operator $U(t) = e^{it\sqrt{\Delta}}$ at times t in the length spectrum of (M, g) (including $t = 0$).

Heat invariants

The heat trace

$$Tre^{-t\Delta} = \sum_{j=0}^{\infty} e^{-\lambda_j^2 t}$$

on a compact Riemannian manifold without boundary has the asymptotic expansion,

$$Tre^{t\Delta_g} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j \ t^j. \qquad (2)$$

The coefficients a_j are the heat invariants. When n is odd, the powers of t are singular and hence the expansion may be viewed as a singularity expansion in which the terms become more regular. When n is even, the terms with $-n/2 + j < 0$ are singular but the rest are smooth and hence are not residual.

The first four heat invariants

The first four heat invariants in the boundaryless case are given by

$$
\begin{aligned}
a_0 &= Vol(M) = \int dVol_M \\
a_1 &= \frac{1}{6}\int S dVol_M \\
a_2 &= \frac{1}{360}\int \{2|R|^2 \ - 2|Ric|^2 \ + 5S^2] dVol_M \\
a_3 &= \frac{1}{6!}\int \{-\frac{1}{9}|\nabla R|^2 - \frac{26}{63}|\nabla Ric|^2 - \frac{143}{63}|\nabla S|^2 \\
&-\frac{8}{21}R^{ij}_{k\ell}R^{kl}_{rs}R^{rs}_{ijk\ell} - \frac{8}{63}R^{rs}R^{jk\ell}_{r}R_{sjk\ell} + \frac{2}{3}S|R|^2 \\
&-\frac{20}{63}R^{ik}R_{j\ell}R_{ijk\ell} - \frac{4}{7}R^i_jR^j_kR^k_i - \frac{2}{3}S|Ric|^2 + \frac{5}{9}S^3\} dVol_M.
\end{aligned}
\tag{3}
$$

Here, S is the scalar curvature, Ric is the Ricci tensor and R is the Riemann tensor. In general the heat invariants are integrals of curvature polynomials of various weights in the metric.

Applications of heat invariants

- Spheres: Tanno used a_0, a_1, a_2, a_3 to prove that the round metric g_0 on S^n for $n \leq 6$ is determined among all Riemannian manifolds by its spectrum, i.e. any isospectral metric g is necessarily isometric to g_0. He also used a_3 to prove that canonical spheres are locally spectrally determined (hence spectrally rigid) in all dimensions. Patodi proved that round spheres are determined by the spectra $\mathrm{Spec}^0(M, g)$ and $\mathrm{Spec}^1(M, g)$ on zero and 1 forms.
- Same for Complex Projective space $\mathbb{P}^n$ with its Fubini-Study metric.
- Flat manifolds: Patodi and Tanno used heat invariants to prove in dimension ≤ 5 that if (M, g) is isospectral to a flat manifold, then it is flat: they showed that if $a_j = 0$ for $j \geq 1$, and if $n \leq 5$ then (M, g) is flat. Kuwabara used heat invariants to prove that flat manifolds are locally spectrally determined, hence spectrally rigid.

The boundary case

When $\partial\Omega \neq 0$, the heat trace has the form

$$Tr e^{t\Delta_g} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j \, t^{j/2}. \qquad (4)$$

For plane domains, L. Smith obtained the first five heat kernel coefficients in the case of Dirichlet boundary conditions.

$$\begin{cases} a_0 = \text{ area of } \Omega, \\ a_1 = -\sqrt{\frac{\pi}{2}} \, |\partial\Omega| \text{ (the length of the boundary)}, \\ a_2 = \frac{1}{3} \int_{\partial\Omega} \kappa ds, \\ a_3 = \frac{\sqrt{\pi}}{64} \int_{\partial\Omega} \kappa^2 ds, \\ a_4 = \frac{4}{315} \int_{\partial\Omega} \kappa^3 ds, \\ a_5 = \frac{37\sqrt{\pi}}{2^{13}} \int_{\partial\Omega} \kappa^4 ds - \frac{\sqrt{\pi}}{2^{10}} \int_{\partial\Omega} (\kappa')^2 ds, \end{cases} \qquad (5)$$

Applications

In all dimensions, one still has

$$a_0 = C_n\ Vol_n(\Omega), \quad a_1 = C'_n\ Vol_{n-1}(\partial\Omega). \tag{6}$$

Hence:

- Euclidean balls in all dimensions are spectrally determined among simply connected bounded Euclidean domains by their Dirichlet or Neumann spectra. This follows from (6) and from the fact that isoperimetric hypersurfaces in $\mathbb{R}^n$ are spheres.

Domains and metrics with the same heat invariants

Heat invariants are insufficient to determine smooth metrics or domains. This is due to the fact that they are integrals of local invariants of the metrics. Pairs of non-isometric metrics with the same heat invariants can be obtained by putting two isometric bumps. The bumped spheres will not be isometric if the distances between the bumps are different, but the heat invariants will be the same. There are many variations on this well-known example. But heat invariants might be quite useful for analytic metrics and domains, and have also been used in compactness results.

Wave invariants

These are the principal invariants we discuss from now on. The wave group is the unitary group $U(t) = e^{it\sqrt{\Delta}}$ of (M, g). The wave trace is the (distribution) trace

$$TrU(t) = \sum_{\lambda_j \in Sp(\sqrt{\Delta})} e^{it\lambda_j}. \tag{7}$$

It is a tempered distribution on $\mathbb{R}$. We denote its singular support (the complement of the set where it is a smooth function) by Sing Supp $TrU(t)$.

Poisson relation and length spectrum

The first result on the wave trace is the Poisson relation on a manifold without boundary,

$$\text{Sing Supp}\, TrU(t) \subset \; Lsp(M, g), \tag{8}$$

where $Lsp(M, g)$ is the *length spectrum* of (M, g), i.e. the set of lengths of closed geodesics. (Colin de Verdière, Chazarain, and Duistermaat-Guillemin, following physicists Balian-Bloch and Gutzwiller).

We denote the length of a closed geodesic γ by L_γ. For each $L = L_\gamma \in Lsp(M, g)$ there are at least two closed geodesics of that length, namely γ and γ^{-1} (its time reversal). The singularities due to these lengths are identical so one often considers the even part of $TrU(t)$ i.e. $TrE(t)$ where $E(t) = \cos(t\sqrt{\Delta})$.

Mulitplicities in the length Spectrum

There may exist two geometrically distinct closed geodesics of the same length. $Lsp(M, g)$ does not include information about multiplicities of lengths. This is the main obstacle to using the wave trace in inverse spectral theory.

Singularity expansions

Much more is true than the Poisson relation: $TrU(t)$ has a singularity expansion at each $L \in Lsp(M, g)$:

$$TrU(t) \equiv e_0(t) + \sum_{L \in Lsp(M,g)} e_L(t) \;\; mod \;\; C^\infty, \qquad (9)$$

where e_0, e_L are Lagrangean distributions with singularities at just one point, i.e. $singsuppe_0 = \{0\}, singsuppe_L = \{L\}$.

At $t = 0$ the wave trace is essentially equivalent to the heat trace:

$$e_0(t) = a_{0,-n}(t + i0)^{-n} + a_{0,-n+1}(t + i0)^{-n+1} + \cdots \qquad (10)$$

The wave coefficients $a_{0,k}$ at $t = 0$ are essentially the same as the singular heat coefficients, hence are given by integrals over M of $\int_M P_j(R, \nabla R, ...)$dvol of homogeneous curvature polynomials.

Non-degenerate closed geodesics

In the *non-degenerate* case (defined below),

$$\begin{aligned} e_L(t) &= a_{L,-1}(t - L + i0)^{-1} + a_{L,0} \log(t - (L + i0)) \\ &+ a_{L,1}(t - L + i0) \log(t - (L + i0)) + \cdots \quad , \end{aligned} \tag{11}$$

where $\cdots$ refers to homogeneous terms of ever higher integral degrees. The wave invariants for $t \neq 0$ have the form:

$$a_{L,j} = \sum_{\gamma : L_\gamma = L} a_{\gamma,j}, \tag{12}$$

where $a_{\gamma,j}$ involves on the germ of the metric along γ. Here, $\{\gamma\}$ runs over the set of closed geodesics of length L; L_γ, resp. $L_\gamma^\#$, are the length, resp. primitive length of γ.

Principal wave invariant at γ

The principal wave invariant at $t = L$ in the case of a non-degenerate closed geodesic is given by

$$a_{L,-1} = \sum_{\gamma : L_\gamma = L} \frac{e^{\frac{i\pi}{4} m_\gamma} L_\gamma^{\#}}{|\det(I - P_\gamma)|^{\frac{1}{2}}}. \tag{13}$$

Here, m_γ, resp. P_γ denote the Maslov index and linear Poincaré map of γ.

The same formula for the leading singularity is valid for periodic reflecting rays of compact smooth Riemannian domains with boundary and with Neumann boundary conditions, while in the Dirichlet case the numerator must be multiplied by $(-1)^r$ where r is the number of reflection points.

Comparison of heat and wave invariants

The wave invariants for $t \neq 0$ are both less global and more global than the heat invariants. First, they are more global in that they are not integrals of local invariants, but involve the semi-global first return map $\mathcal{P}_\gamma$. One could imagine different local geometries producing the same first return map. Second, they are less global because they are determined by the germ of the metric at γ and are unchanged if the metric is changed outside γ.

Wave invariants of iterates γ^r

Thus, associated to any closed geodesic γ of (M, g) is the sequence $\{a_{\gamma^r,j}\}$ of wave invariants of γ and of its iterates γ^r. These invariants depend only on the germ of the metric at γ. The principal question of this survey may be stated as follows:

How much of the local geometry of the metric g at γ is contained in the wave invariants $\{a_{\gamma^r,j}\}$? Can the germ of the metric g at γ be determined from the wave invariants? At least, can the symplectic equivalence class of its germ be determined?

Wave trace and dynamics of the geodesic flow

Unlike heat invariants, the wave invariants $a_{\gamma,j}$ involve a mixture of Riemannian and symplectic invariants of the geodesic flow.

We denote by $(T^*M, \sum_j dx_j \wedge d\xi_j)$ the cotangent bundle of M equipped with its natural symplectic form. Given a metric g, we define the metric Hamiltonian by

$$H(x,\xi) = |\xi| := \sqrt{\sum_{ij=1}^{n+1} g^{ij}(x)\xi_i\xi_j} \tag{14}$$

Let

$$\Xi_H = \text{ the Hamiltonian vector field of } H.$$

The geodesic flow is the Hamiltonian flow

$$G^t = \exp t\Xi_H : T^*M\backslash \to T^*M\backslash 0.$$

It is homogeneous of degree 1 with respect to the dilation $(x,\xi) \to (x, r\xi), r > 0$.

Restriction to $S_g^* M$

The energy surface is the unit co-sphere bundle

$$S_g^* M = \{(x, \xi)| : H(x, \xi) = |\xi|_g = 1\}.$$

It is invariant under G^t and we usually refer to the geodesic flow as

$$G^t : S_g^* M \to S_g^* M.$$

Since G^t is homogeneous, nothing is lost by restricting G^t to $S_g^* M$.

Closed orbits and their Poincaré maps

Closed orbits (or periodic orbits) γ of flows are orbits of points $(x, \xi) \in T^*M$ satisfying $G^T(x, \xi) = (x, \xi)$ for some $T \neq 0$ (the period). They project to closed geodesics on the Riemannian manifold or domain.

We recall the definition of the nonlinear Poincare map $\mathcal{P}_\gamma$: in S^*M one forms a symplectic transversal S_γ to γ at some point m_0. One then defines the first return map, or nonlinear Poincarë map,

$$\mathcal{P}_\gamma(\zeta) : S_\gamma \to S_\gamma$$

by setting $\mathcal{P}_\gamma(\zeta) = G^{T(\zeta)}(\zeta)$, where $T(\zeta)$ is the first return time of the trajectory to S_γ. This map is well-defined and symplectic from a small neighborhood of $\gamma(0) = m_0$ to a larger neighborhood. By definition, the linear Poincare map is its derivative, $P_\gamma = d\mathcal{P}_\gamma(m_0)$.

Linear Poincaré map

We let $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ denote the space of complex normal Jacobi fields along γ, a symplectic vector space of (complex) dimension $2n$ ($n = \dim M - 1$) with respect to the Wronskian

$$\omega(X, Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincaré map P_γ is then the linear symplectic map on $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ defined by $P_\gamma Y(t) = Y(t + L_\gamma)$.

Types of closed geodesics

Closed geodesics are classified by the spectral properties of the symplectic linear map P_γ. Its eigenvalues come in 4-tuples $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$. A closed geodesic γ is called:

- *non-degenerate* if $\det(I - P_\gamma) \neq 0$;
- *elliptic* if all of its eigenvalues are of modulus one and not equal to ± 1, in which case they come in complex conjugate pairs $e^{i \pm \alpha_j}$.
- *hyperbolic* if all of its eigenvalues are real, in which case they come in inverse pairs $\lambda_j \lambda_j^{-1}$
- *loxodromic* or *complex hyperbolic* in the case where the 4-tuple consists of distinct eigenvlaues as above.

Degenerate periodic orbits

In the notation for $Lsp(M, g)$ we wrote L_{γ_j} as if the closed geodesics of this length were isolated. But in many examples (e.g. spheres or flat tori), the geodesics come in families, and the associated length T is the common length of closed geodesics in the family. In place of closed geodesics, one has components of the fixed point sets of G^T at this time. The fixed point sets could be quite messy, so it is also common to assume that they are clean, i.e. that the fixed point sets are manifolds, and that their tangent spaces are fixed point sets of dG^T. It is equivalent that the length functional is Bott-Morse on the free loop space.

There are other possibilities (parabolic) in the degenerate case. For instance, the geodesics of a flat torus of dimension n come in n-parameter families. ♣ Integrable. ♣

The length spectrum and the marked length spectrum

The *length spectrum* of a boundaryless manifold (M, g) is the discrete set

$$Lsp(M, g) = \{L_{\gamma_1} < L_{\gamma_2} < \cdots\} \tag{15}$$

of lengths of closed geodesics γ_j.