CIRM Summer Preschool on Inverse Problems

Boundary Rigidity

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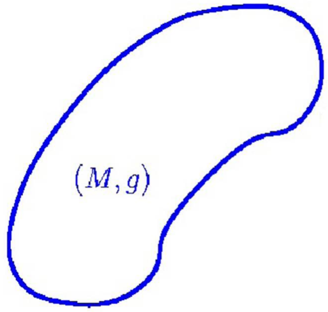


(M,g) Riemannian manifold with boundary, $g = (g_{ij})$ symmetric positive definite matrix.

Can one recover the geometry and topology from boundary measurements?

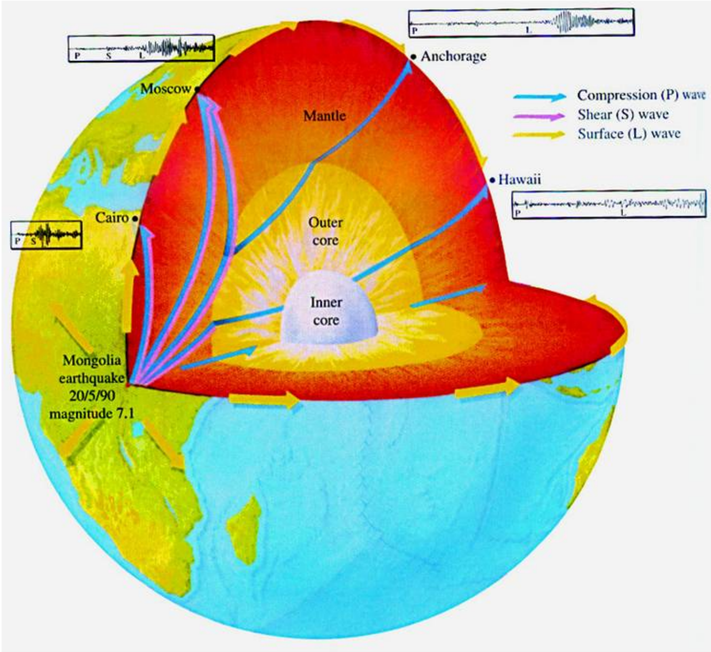
Two types of measurements:

(1) Boundary distance function or scattering relation, (travel time)

(2) Dirichlet to Neumann map from Laplace-Beltrami operator, (Calderón Problem, EIT)

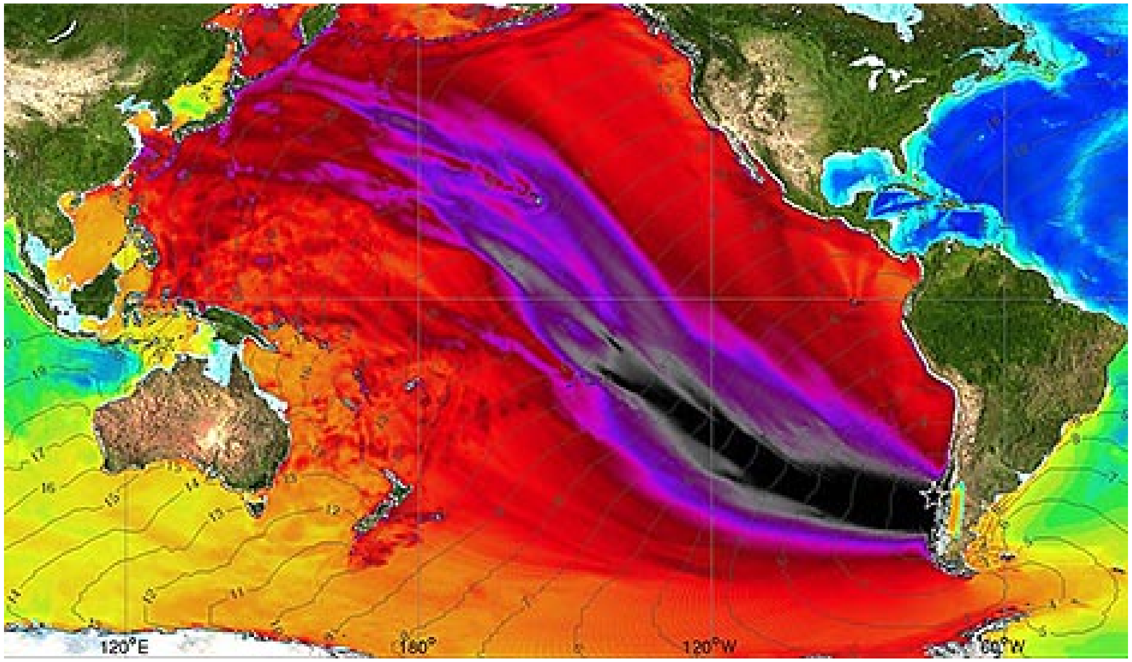
Travel Time Tomography (Transmission)

Global Seismology



Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

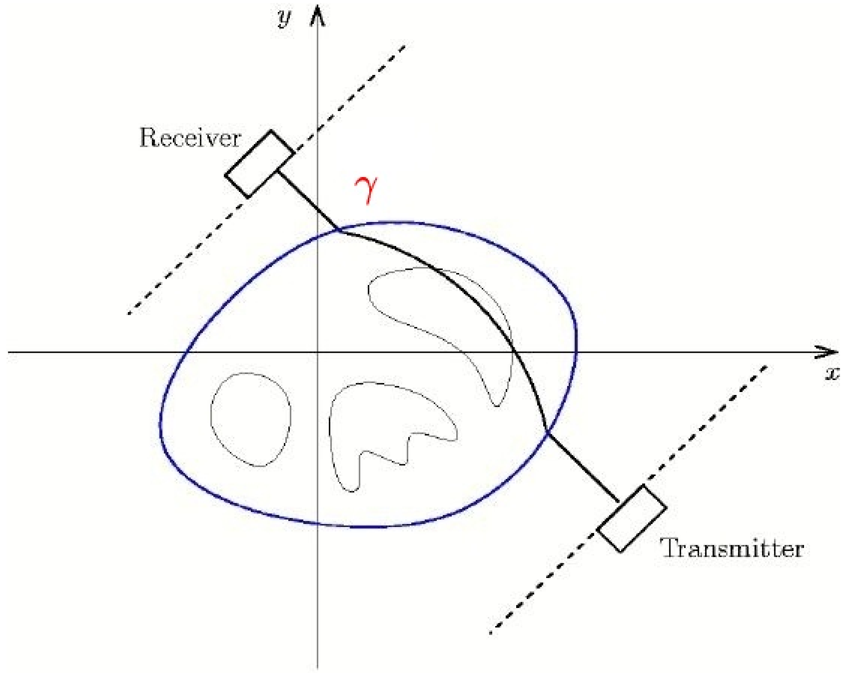
Tsunami of 1960 Chilean Earthquake



Black represents the largest waves, decreasing in height through purple, dark red, orange and on down to yellow. In 1960 a tongue of massive waves spread across the Pacific, with big ones throughout the region.

Human Body Seismology

ULTRASOUND TRANSMISSION TOMOGRAPHY(UTT)

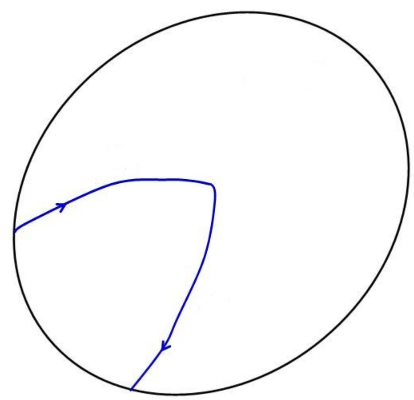


$$T = \int_{\gamma} \frac{1}{c(x)}\, ds$$ = Travel Time (Time of Flight).

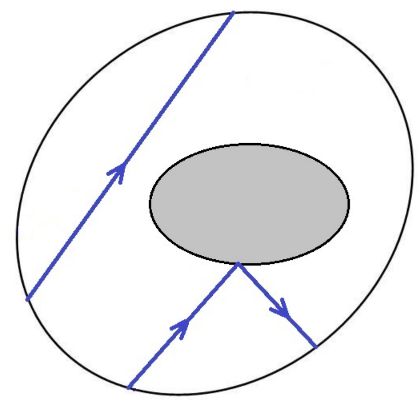
REFLECTION TOMOGRAPHY

Scattering

Points in medium

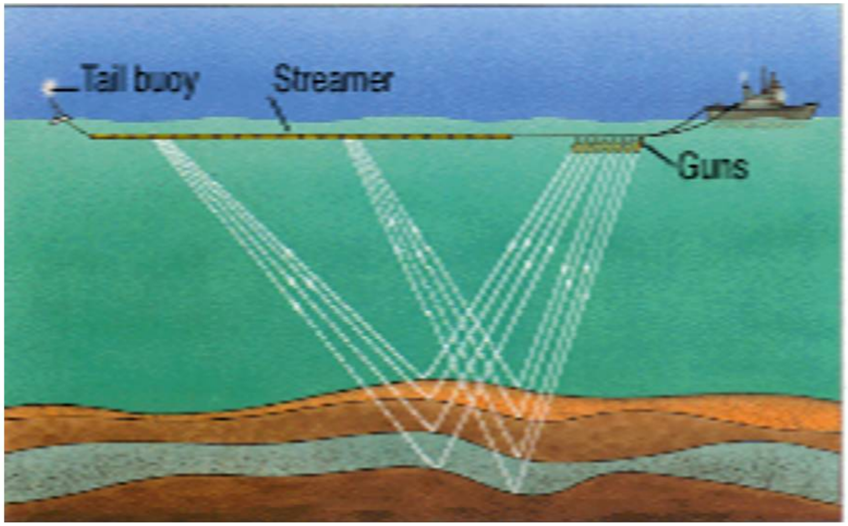


Obstacle



REFLECTION TOMOGRAPHY

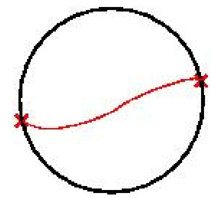
Oil Exploration



Ultrasound

TRAVELTIME TOMOGRAPHY (Transmission)

Motivation:Determine inner structure of Earth by measuring travel times of seismic waves



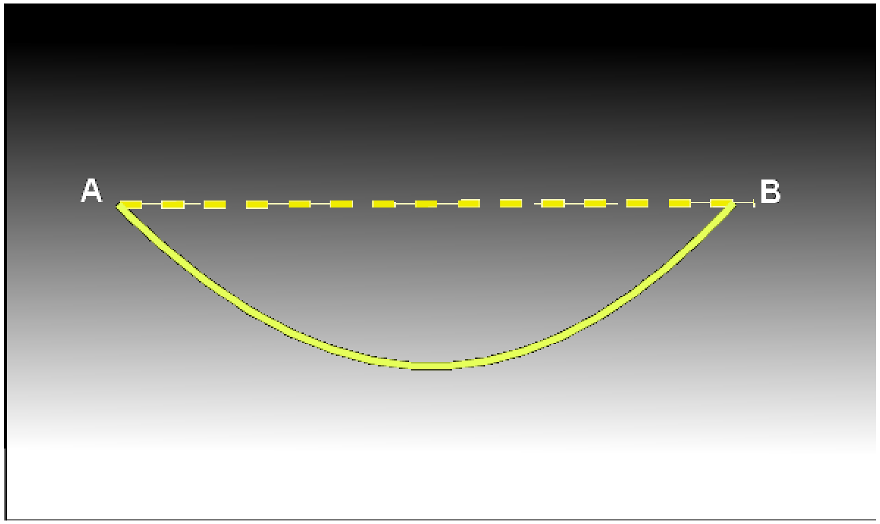
Herglotz, Wiechert-Zoeppritz (1905)

Sound speed $c(r)$, $r = |x|$

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

$T = \int_\gamma \frac{1}{c(r)}$. What are the curves of propagation γ?

Ray Theory of Light: Fermat's principle



Fermat's principle. Light takes the shortest optical path from A to B (solid line) which is not a straight line (dotted line) in general. The optical path length is measured in terms of the refractive index n integrated along the trajectory. The greylevel of the background indicates the refractive index; darker tones correspond to higher refractive indices.

The curves are geodesics of a metric.

$$ds^2 = \frac{1}{c^2(r)} dx^2$$

More generally $ds^2 = \frac{1}{c^2(x)} dx^2$

Velocity $v(x, \xi) = c(x), \quad |\xi| = 1$ (isotropic)

Anisotropic case

$$\boxed{ds^2 = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j}$$

$g = (g_{ij})$ is a positive definite symmetric matrix

Velocity $v(x, \xi) = \sqrt{\sum_{i,j=1}^{n} g^{ij}(x) \xi_i \xi_j}, \quad |\xi| = 1$

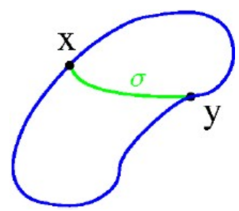
$$g^{ij} = (g_{ij})^{-1}$$

The information is encoded in the boundary distance function

More general set-up

(M, g) a Riemannian manifold with boundary

(compact) $g = (g_{ij})$



$x, y \in \partial M$

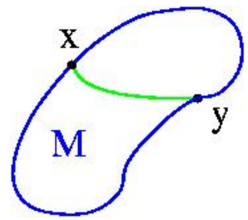
$$d_g(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

$L(\sigma)$ = length of curve σ

$$L(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt}\frac{d\sigma_j}{dt}} dt$$

Inverse problem

Determine g knowing $d_g(x,y)$ $x, y \in \partial M$



$dg \Rightarrow g$?

(Boundary rigidity problem)

Answer NO $\psi : M \to M$ diffeomorphism

$$\psi\Big|_{\partial M} = \text{Identity}$$

$$d_{\psi^* g} = d_g$$

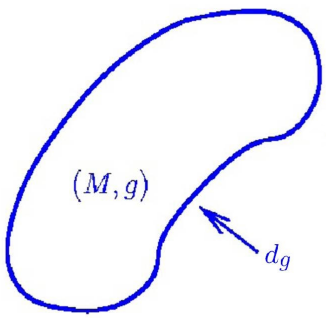
$$\psi^* g = \left(D\psi \circ g \circ (D\psi)^T \right) \circ \psi$$

$$L_g(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

$$\tilde{\sigma} = \psi \circ \sigma \quad L_{\psi^* g}(\tilde{\sigma}) = L_g(\sigma)$$

ANOTHER MOTIVATION (STRING THEORY)

HOLOGRAPHY

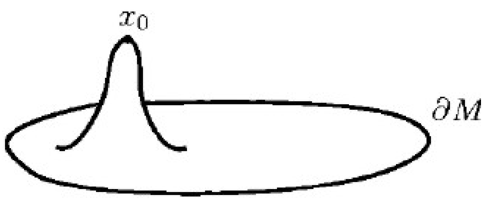


Inverse problem: Can we recover (M, g) (bulk) from boundary distance function ?

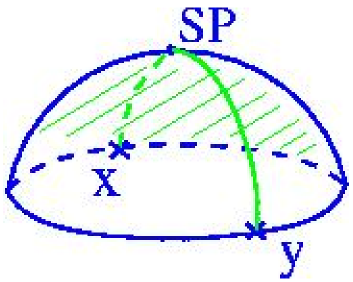
M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 0401 (2004) 034

$$\boxed{d_{\psi^* g} = d_g}$$

Only obstruction to determining g from d_g ? No



$$d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x,y)$$



Can change metric near SP

Def (M, g) is boundary rigid if $(M, \tilde{g})$ satisfies $d_{\tilde{g}} = d_g$. Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} =$ Identity, so that

$$\boxed{\tilde{g} = \psi^* g}$$

Need an a-priori condition for (M, g) to be boundary rigid.

One such condition is that (M, g) is simple

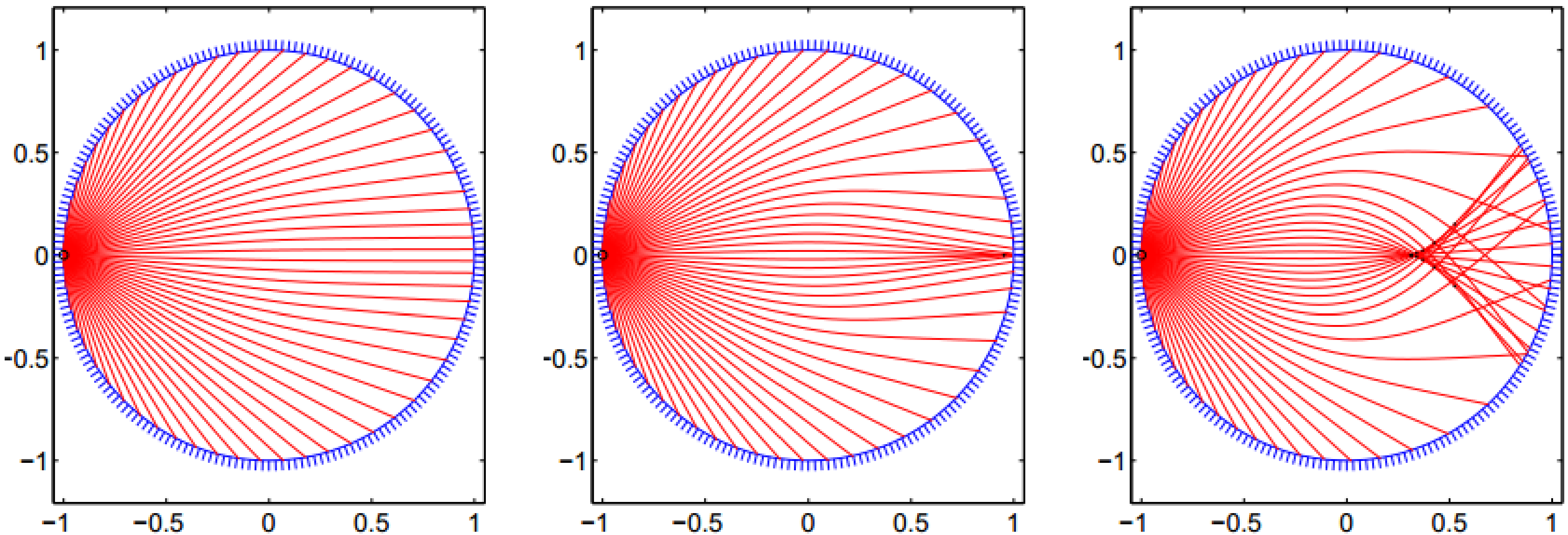
<u>DEF</u> (M, g) is simple if given two points $x, y \in \partial M$, $\exists$! geodesic joining x and y and ∂M is strictly convex

CONJECTURE

(M, g) is simple then (M, g) is boundary rigid ,that is $\boxed{d_g}$ determines $\boxed{g}$ up to the natural obstruction.
$(d_{\psi^* g} = d_g)$

(Conjecture posed by R. Michel, 1981)

Metrics Satisfying the Herglotz condition



$k = 0.20$ (simple) $k = 0.49$ (non-simple) $k = 1.23$ (non-simple)

$$g_k(r) = \exp\left(k \exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \quad 0 \le r \le 1, \quad \sigma \text{ fixed}$$

Francois Monard: SIAM J. Imaging Sciences (2014)

Results for Isotropic Case

$$d_{\beta g} = d_g \Longrightarrow \beta = 1?$$

<u>Theorem</u> (Mukhometov, Mukhometov-Romanov, Beylkin, Gerver-Nadirashvili, ...)

YES for simple manifolds.

The sound speed case corresponds to $g = \frac{1}{c^2} e$ with e the identity.

Results (M, g) simple

- R. Michel (1981) Compact subdomains of $\mathbb{R}^2$ or $\mathbb{H}^2$ or the open round hemisphere

- Gromov (1983) Compact subdomains of $\mathbb{R}^n$

- Besson-Courtois-Gallot (1995) Compact subdomains of negatively curved symmetric spaces

(All examples above have constant curvature)

- $\left.\begin{array}{l}\text{Stefanov-U (1998)} \\ \text{Lassas-Sharafutdinov-U (2003)} \\ \text{Burago-Ivanov (2010)}\end{array}\right\}$ $\boxed{dg = dg_0}$, g_0 close to Euclidean

$n = 2$

- Otal and Croke (1990) $K_g < 0$

THEOREM(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

Theorem ($n \geq 3$) (Stefanov-U, 2005)

There exists a generic set $\widetilde{\mathcal{L}} \subset C^k(M) \times C^k(M)$ such that

$$(g_1, g_2) \in \widetilde{\mathcal{L}},\ g_i \text{ simple},\ i = 1, 2,\ d_{g_1} = d_{g_2}$$

$$\Longrightarrow \exists \psi :\ M \to M \text{ diffeomorphism},$$

$$\psi\Big|_{\partial M} = \text{Identity, so that } \boxed{g_1 = \psi^* g_2}.$$

Remark

If M is an open set of $\mathbb{R}^n$, $\widetilde{\mathcal{L}}$ contains all pairs of simple and real-analytic metrics in $C^k(M)$.

Theorem ($n \geq 3$) (Stefanov-U, 2005)

(M, g_i) simple $i = 1, 2$, g_i close to $g_0 \in \mathcal{L}$ where $\mathcal{L}$ is a generic set of simple metrics in $C^k(M)$. Then

$$d_{g_1} = d_{g_2} \Rightarrow \exists \psi : \ M \to M \text{ diffeomorphism,}$$

$$\psi\Big|_{\partial M} = \text{Identity, so that } \boxed{g_1 = \psi^* g_2}$$

Remark

If M is an open set of $\mathbb{R}^n$, $\mathcal{L}$ contains all simple and real-analytic metrics in $C^k(M)$.

Geodesics in Phase Space

$g = \big(g_{ij}(x)\big)$ symmetric, positive definite

Hamiltonian is given by

$$H_g(x,\xi) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j - 1\Big) \qquad g^{-1} = \big(g^{ij}(x)\big)$$

$X_g(s,X^0) = \big(x_g(s,X^0), \xi_g(s,X^0)\big)$ be bicharacteristics,

sol. of $$\frac{dx}{ds} = \frac{\partial H_g}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_g}{\partial x}$$

$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_g^{n-1}(x^0)$

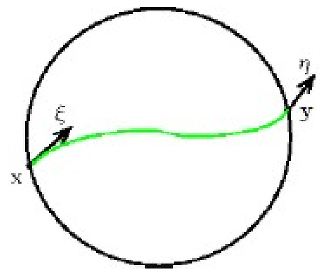
$$\mathcal{S}_g^{n-1}(x) = \Big\{\xi \in \mathbb{R}^n;\ H_g(x,\xi) = 0\Big\}.$$

Geodesics Projections in x: $x(s)$.

Scattering Relation

d_g only measures first arrival times of waves.

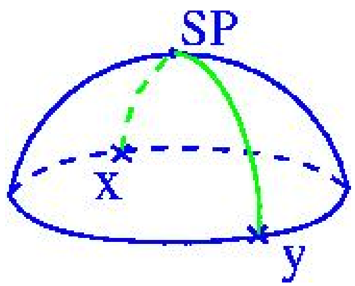
We need to look at behavior of all geodesics



$$\|\xi\|_g = \|\eta\|_g = 1$$

$\alpha_g(x, \xi) = (y, \eta)$, α_g is SCATTERING RELATION

If we know direction and point of entrance of geodesic then we know its direction and point of exit.



Scattering relation follows all geodesics.

Conjecture Assume (M,g) non-trapping. Then α_g determines g up to natural obstruction.

(Pestov-U, 2005) $n = 2$ Connection between α_g and Λ_g (Dirichlet-to-Neumann map)

(M, g) simple then $d_g \Leftrightarrow \alpha_g$

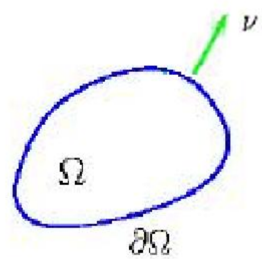
Dirichlet-to-Neumann Map (Lee–U, 1989)

(M, g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j} \right) \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\Delta_g u = 0 \text{ on } M, \qquad u\big|_{\partial M} = f$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g} \Big|_{\partial M}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

$$\Delta_g u = 0$$
$$u\big|_{\partial M} = f$$

$$\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}\Big|_{\partial M}$$

$$\Lambda_g \Rightarrow g \quad ?$$

Answer: No

$$\Lambda_{\psi^* g} = \Lambda_g$$

$$Q_g(f) = \sum_{i,j} \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \sqrt{\det g}\, dx$$

$$Q_g(f) = \int_{\partial M} \Lambda_g(f) f \, dS$$

$$Q_g \Leftrightarrow \Lambda_g$$

$\psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity

$v = u \circ \psi$, $\Delta_{\psi^* g} v = 0$

$$Q_{\psi^* g} = Q_g \Rightarrow \Lambda_{\psi^* g} = \Lambda_g$$

$$Q_g(f) = \sum_{i,j} \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \sqrt{\det g}\, dx$$

$Q_{\psi^* g}(f) = Q_g(f)$ $\psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity

$$\boxed{\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi}$$

$$\boxed{\Lambda_{\psi^* g} = \Lambda_g}$$

Problem: Only obstruction?

$n = 2$, Additional obstruction

$$\Delta_{\beta(x) g} u = \frac{1}{\beta(x)} \Delta_g u, \beta > 0.$$

$$\Lambda_{\beta(x) g} f = \Lambda_g f \text{ if } \beta\big|_{\partial M} = 1$$

In $n = 2$ these are the only obstructions

Theorem ($n = 2$)(Lassas-U, 2001)

(M, g_i), $i = 1, 2$, connected Riemannian manifold with boundary. Assume

$$\Lambda_{g_1} = \Lambda_{g_2}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity, and $\beta > 0, \beta\big|_{\partial M} = 1$ so that

$$g_1 = \beta \psi^* g_2$$

In fact, one can determine topology of M as well.

Theorem ($n \geq 3$) (Lassas-U 2001, Lassas-Taylor-U 2003) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity, so that

$$\boxed{g_1 = \psi^* g_2}$$

One can also determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Theorem (Guillarmou-Sa Barreto, 2009) $(M, g_i), i = 1, 2$, are compact Riemannian manifolds with boundary that are Einstein. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi|_{\partial M} = Identity$ such that

$$\boxed{g_1 = \psi^* g_2}$$

Note: Einstein manifolds with boundary are real analytic in the interior.

$\boxed{n = 2}$

<u>THEOREM</u>(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

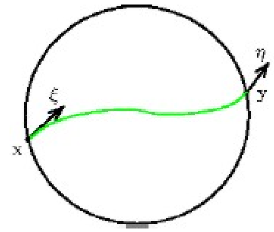
CONNECTION BETWEEN BOUNDARY RIGIDITY AND DIRICHLET-TO-NEUMANN MAP

THEOREM ($n = 2$) (Pestov-U, 2005)

If we know d_g then we can determine Λ_g if (M, g) simple.

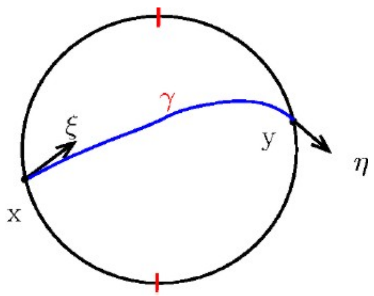
IN FACT (M, g) simple $n = 2$

$$d_g \Rightarrow \alpha_g \Rightarrow \Lambda_g$$



$$\alpha_g(x, \xi) = (y, \eta)$$

CONNECTION BETWEEN SCATTERING RELATION AND DIRICHLET-TO-NEUMANN MAP$(n = 2)$



$$\alpha_g(x, \xi) = (y, \eta)$$

$\boxed{\alpha_g}$ determines $\boxed{\Lambda_g}$ if I^* is onto.

$$\boxed{If(x, \xi) = \int_\gamma f}$$

I^* is onto if I is injective for simple manifolds

Now $\Lambda_g \overset{L-U}{\Longrightarrow} \beta\psi^* g, \ \beta > 0$

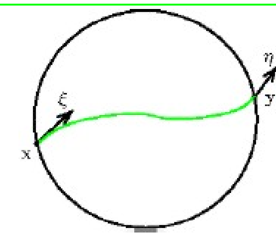
If I is injective, we can also recover β for simple manifolds.

Dirichlet-to-Neumann map	Boundary distance function
$\Lambda_g(f)(x) = \int_{\partial M} \lambda_g(x,y) f(y) dS_y$ λ_g depends on 2n-2 variables	$d_g(x,y), \quad x,y \in \partial M$ $d_g(x,y)$ dep. on 2n-2 variables
$\Delta_g u = 0, \quad u\big\|_{\partial M} = f$ $\Lambda_g \Longleftrightarrow Q_g$ $Q_g(f) = \sum \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$ $= \inf_{v\big\|_{\partial M} = f} \int_M g^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx$	$d_g(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L_g(\sigma)$ $L_g(\sigma) = \int_0^1 \sqrt{g_{ij}(\sigma(t)) \frac{\partial \sigma_i}{\partial t} \frac{\partial \sigma_j}{\partial t}} dt$

Dirichlet-to-Neumann map

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$
$$\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}$$

(Scattering relation)



$$H_g(x,\xi) = \frac{1}{2}\left(\sum g^{ij}\xi_i\xi_j - 1\right)$$
$$\frac{dx_g}{ds} = +\frac{\partial H_g}{\partial \xi}$$
$$\frac{d\xi_g}{ds} = -\frac{\partial H_g}{\partial x}$$
$$x_g(0) = x, \quad \xi_g(0) = \xi, \quad \|\xi\|g = 1$$

we know $(x_g(T), \xi_g(T))$

$$\left\{(f, \Lambda_g(f))\right\} \subseteq L^2(\partial M) \times L^2(\partial M)$$

is Lagrangian manifold

g=e=Euclidean

$$\langle (f_1, g_1), (f_2, g_2)\rangle$$
$$= \int_{\partial M}(g_1 f_2 - f_1 g_2)dS$$

$$\alpha_g(x,\xi) = (y,\eta)$$

$\left\{(x,\xi), \alpha_g(x,\xi)\right\}$ projected to $T^*(\partial M) \times T^*(\partial M)$ is Lagrangian manifold

Theorem (n=2). (M, g_i) simple, $i = 1, 2$.

$$d_{g_1} = d_{g_2} \Rightarrow g_1 = \psi^* g_2,$$

$\psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} = Identity$.

Proof:

$$d_{g_1} = d_{g_2} \Rightarrow \Lambda_{g_1} = \Lambda_{g_2}.$$

$\Lambda_{g_1} = \Lambda_{g_2}. \overset{Lassas-U}{\Longrightarrow} \exists \psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} = Identity$, and $\beta > 0$, $\beta\Big|_{\partial M = 1}$ such that

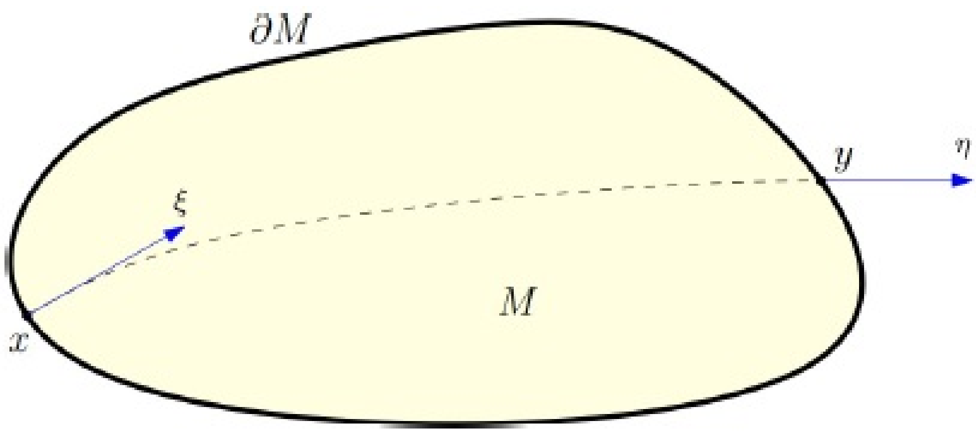
$$g_1 = \beta \psi^* g_2$$

$$d_{g_1} = d_{g_2} = d_{\psi^* g_2} = d_{\frac{1}{\beta} g_1}.$$

Mukhometov $\Rightarrow \beta = 1$. □

Lens Rigidity

Define the scattering relation α_g and the length (travel time) function ℓ:



$$\alpha_g : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change L, ℓ!

Lens rigidity: *Do α_g, ℓ determine g uniquely, up to isometry?*

Lens rigidity: *Do α_g, ℓ determine g uniquely, up to isometry?*

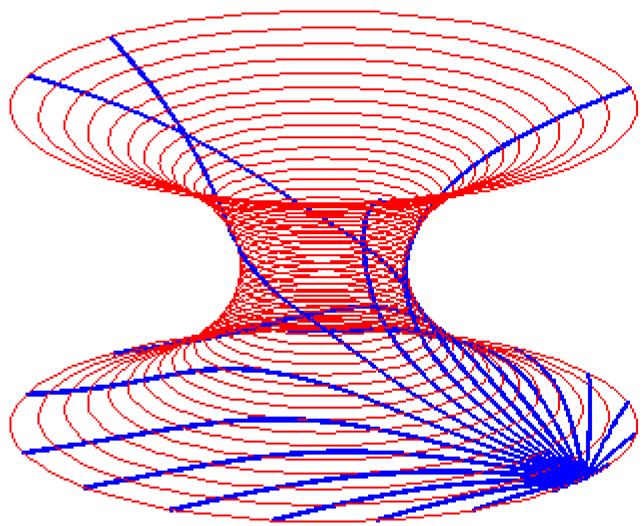
No, There are counterexamples for trapping manifolds (Croke-Kleiner).

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! This is also true locally, near a point p where ∂M is strictly convex.

For non-simple metrics (caustics and/or non-convex boundary), the Lens Rigidity is the right problem to study.

There are fewer results: local generic rigidity near a class of non-simple metrics (Stefanov-U, 2009), for real-analytic metrics satisfying a mild condition (Vargo, 2010), the torus is lens rigid (Croke 2014), stability estimates for a class of non-simple metrics (Bao-Zhang 2014; Stefanov-U-Vasy, 2014; Guillarmou, 2015).

Theorem (C. Guillarmou 2015)**.** Let (M, g) be a surface with strictly convex boundary hyperbolic trapping and no conjugate points. Then lens data determines the metric up to a conformal factor.



Theorem (Vargo, 2009)

(M_i, g_i), $i = 1, 2$, compact Riemannian real-analytic manifolds with boundary satisfying a mild condition. Assume

$$\boxed{\alpha_{g_1} = \alpha_{g_2}}, \quad \boxed{\ell_{g_1} = \ell_{g_2}}$$

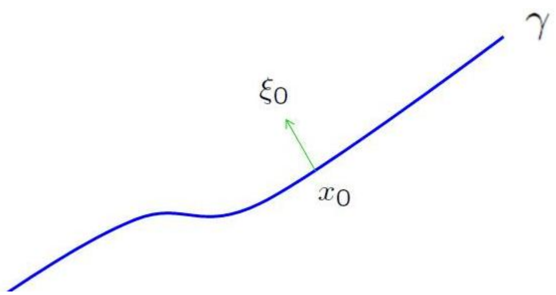
Then $\exists \psi : M \to M$ diffeomorphism, such that

$$\boxed{\psi^* g_1 = g_2}$$

Theorem ($n \geq 3$) (Stefanov-U, 2009)

The lens relation determines the metric near a generic set of metrics under some assumptions on geodesics.

Main one: Given any $(x_0, \xi_0) \in T^*M$, $\exists$ a geodesic γ going through x_0 with normal ξ_0 such that there are no conjugate points on γ.

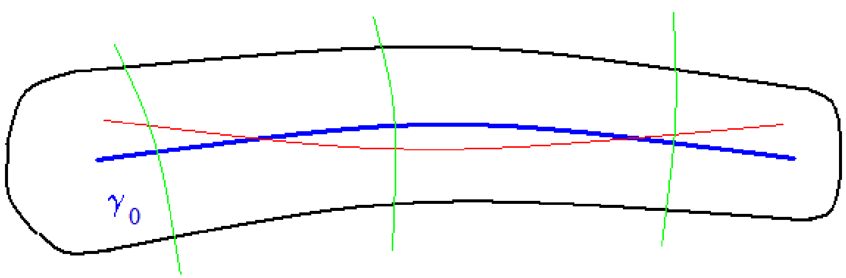


Examples:

A cylinder around an arbitrary geodesic

γ_0 : a finite length geodesic segment on a Riemannian manifold, conjugate points are allowed.

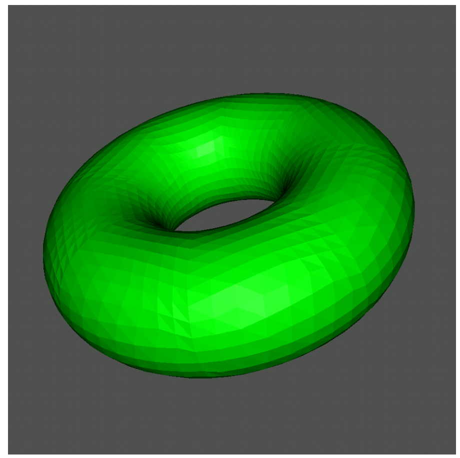
M : a "cylinder" around γ_0, close enough to it.



One can study the scattering relation only for geodesics almost perpendicular to γ_0, there are no conjugate points on them.

The interior of a perturbed torus

$M = S^1 \times \{x_1^2 + x_2^2 \leq 1\}$, g close to the flat one:

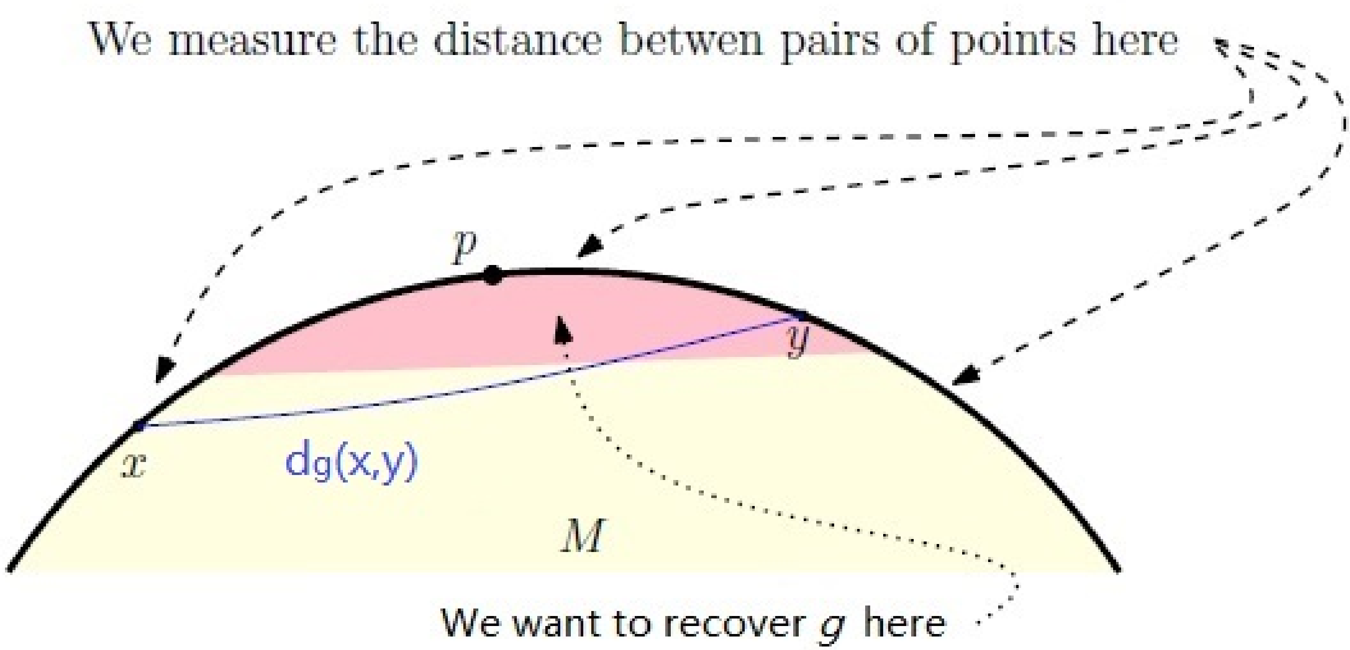


We need only geodesics almost perpendicular to the boundary. Note that M is trapping!

More generally, one can consider a tubular neighborhood of any periodic geodesics on any Riemannian manifold.

Partial Data: General Case

Boundary Rigidity with partial data: Does d_g, known on $\partial M \times \partial M$ near some p, determine g near p up to isometry?



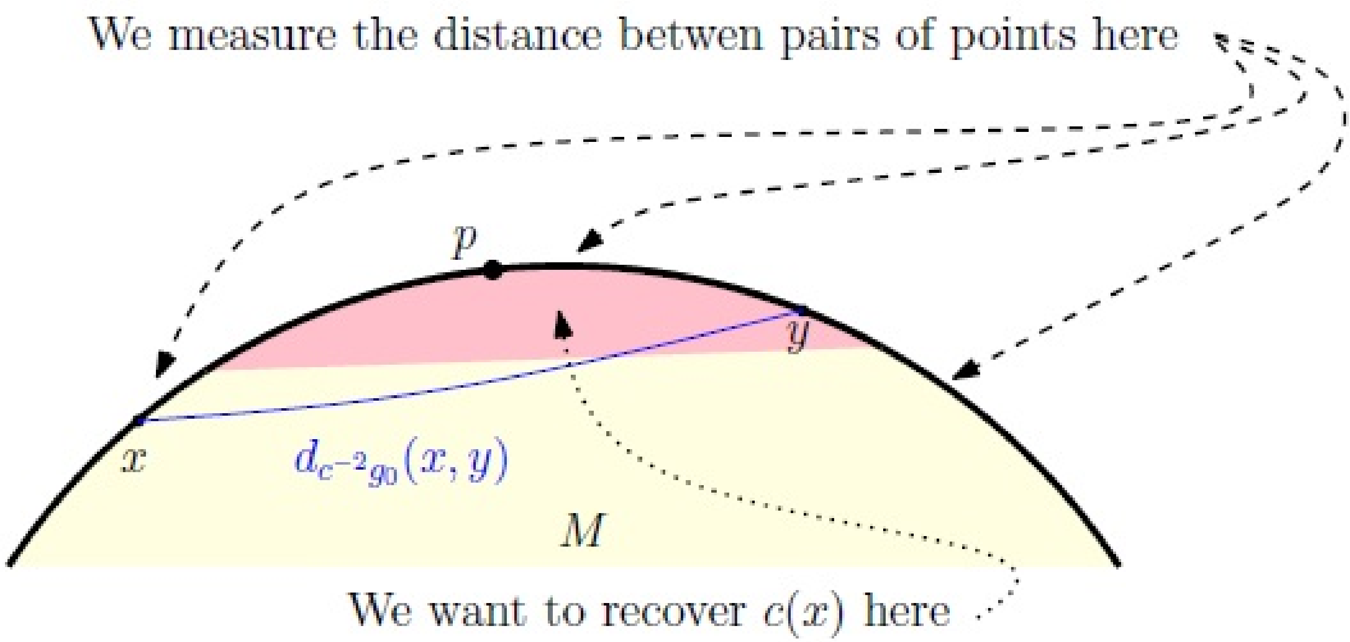
Partial Data: Isotropic Case

Assume that g is isotropic, i.e., $g_{ij}(x) = c^{-2}(x)\delta_{ij}$. Physically, this corresponds to a variable wave speed that does not depend on the direction of propagation. In the class of the isotropic metrics, we do not have the freedom to apply isometries and we would expect g to be uniquely determined.

This is known to be true for simple metrics (Mukhometov, Romanov, et al.) More generally, we can fix g_0 and we have uniqueness of the recovery of the conformal factor $c(x)$ in $c^{-2}g_0$.

Partial Data: Isotropic Case

Boundary Rigidity with partial data: Does $d_{c^{-2}g_0}$, known on $\partial M \times \partial M$ near some p, determine $c(x)$ near p uniquely?



Theorem (Stefanov-U-Vasy, 2013). Let $\dim M \geq 3$. If ∂M is strictly convex near p for c and $\widetilde{c}$, and $d_{c^{-2}g_0} = d_{\widetilde{c}^{-2}g_0}$ near (p, p), then $c = \widetilde{c}$ near p.

Also stability and reconstruction.

The only results so far of similar nature is for real analytic metrics (Lassas-Sharafutdinov-U, 2003). We can recover the whole jet of the metric at ∂M and then use analytic continuation.

Example: Herglotz and Wiechert & Zoeppritz showed that one can determine a radial speed $c(r)$ in the ball $B(0,1)$ satisfying

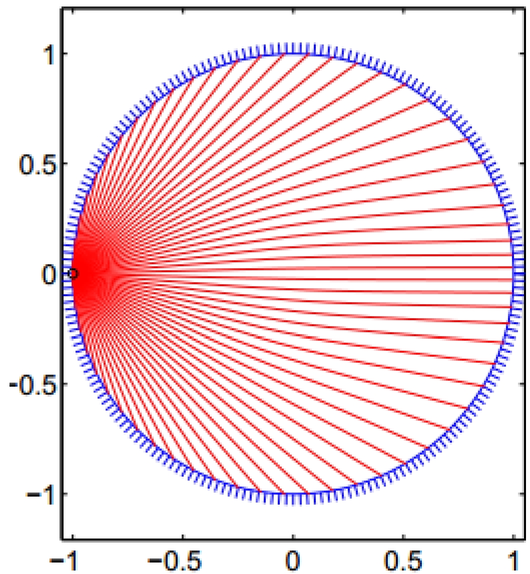
$$\boxed{\frac{d}{dr}\frac{r}{c(r)} > 0.}$$

The uniqueness is in the class of radial speeds.

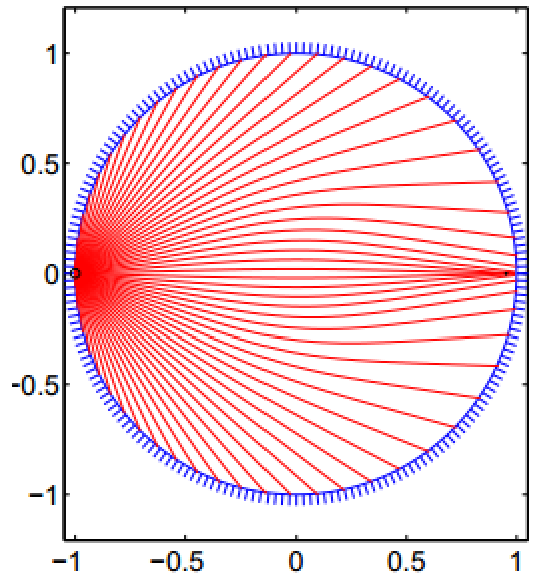
One can check directly that their condition is equivalent to the following one: the Euclidean spheres $\{|x| = t\}$, $t \leq 1$ are strictly convex for $c^{-2}dx^2$ as well. Then $B(0,1)$ satisfies the foliation condition. Therefore, if $\tilde{c}(x)$ is another speed, not necessarily radial, with the same lens relation, equal to c on the boundary, then $c = \tilde{c}$. There could be conjugate points.

Therefore, speeds satisfying the Herglotz and Wiechert & Zoeppritz condition are conformally lens rigid.

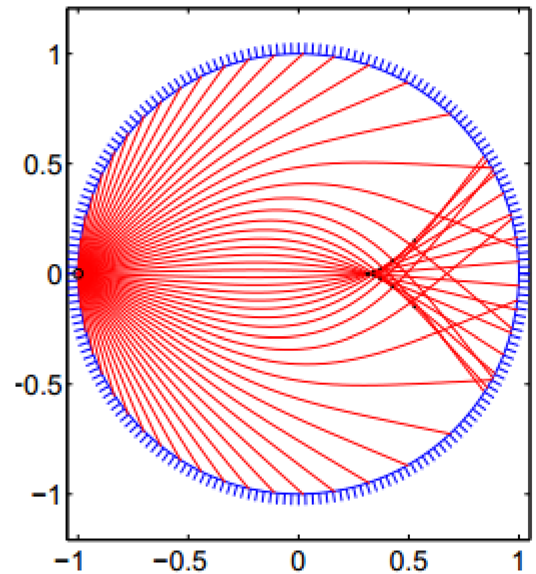
Metrics Satisfying the Herglotz condition



$k = 0.20$ (simple)



$k = 0.49$ (non-simple)



$k = 1.23$ (non-simple)

$$g_k(r) = \exp\left(k \exp\left(-\frac{r^2}{2\sigma^2}\right)\right), \quad 0 \le r \le 1, \quad \sigma \text{ fixed}$$

Francois Monard: SIAM J. Imaging Sciences (2014)

Idea of the proof in isotropic case

The proof is based on two main ideas.

First, we use the approach in a recent paper by U-Vasy (2012) on the linear integral geometry problem.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

First Idea: The Linear Problem

Let (M, g) be compact with smooth boundary. Linearizing $g \mapsto d_g$ in a fixed conformal class leads to the *ray transform*

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt$$

where $x \in \partial M$ and $\xi \in S_x M = \{\xi \in T_x M\,;\, |\xi| = 1\}$.

Here $\gamma(t, x, \xi)$ is the geodesic starting from point x in direction ξ, and $\tau(x, \xi)$ is the time when γ exits M. We assume that (M, g) is *nontrapping*, i.e. τ is always finite.

First Idea: The Linear Problem

U-Vasy result: Consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\,ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". It is proven that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

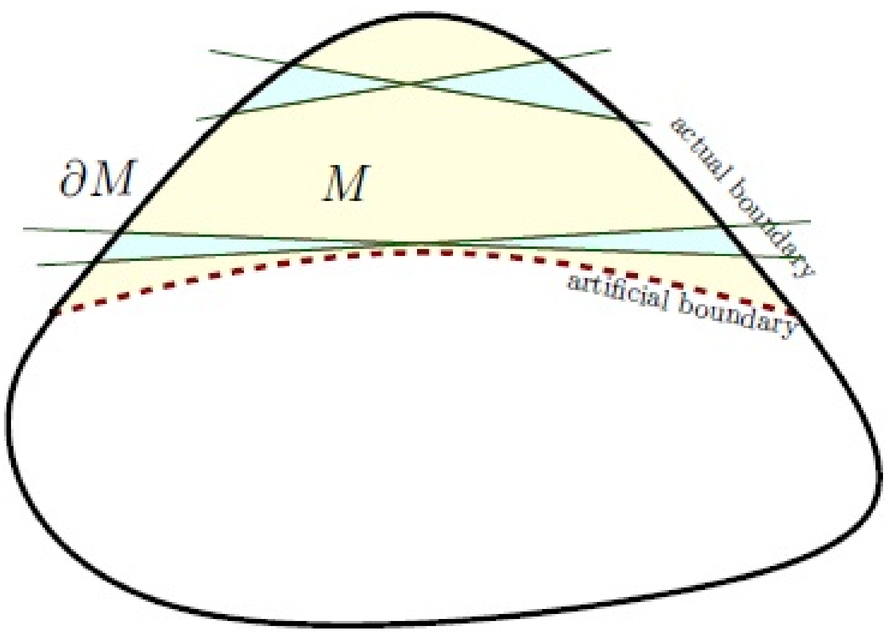
The main idea in U-Vasy is the following:

Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melrose's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

Consider

$$Pf(z) := I^*\chi If(z) = \int_{SM} x^{-2}\chi If(\gamma_{z,v})dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary.



Inversion of local geodesic transform

$$Pf(z) := I^*\chi If(z) = \int_{SM} x^{-2}\chi If(\gamma_{z,v})dv,$$

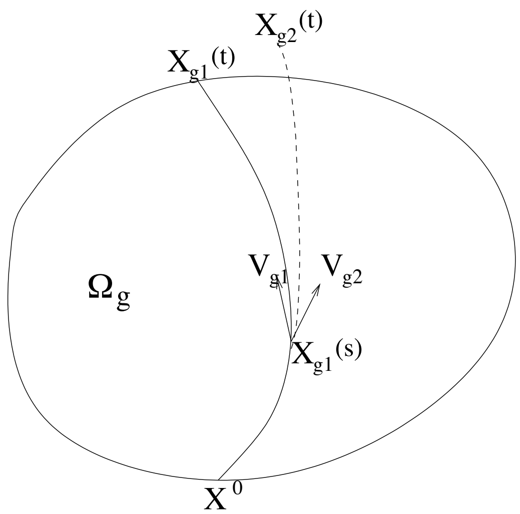
Main result: P is an elliptic pseudodifferential operator in Melrose's scattering calculus.

There exists A such that $AP = Identity + R$

This is Fredholm and R has a small norm in a neighborhood of p. Therefore invertible near p.

Second Step: Reduction to Pseudolinear Problem

Identity (Stefanov-U, 1998)



$$T = d_{g_1},$$
$$F(s) = X_{g_2}\big(T - s, X_{g_1}(s, X^0)\big),$$
$$F(0) = X_{g_2}(T, X^0), \quad F(T) = X_{g_1}(T, X^0),$$
$$\int_0^T F'(s)ds = X_{g_1}(T, X^0) - X_{g_2}(T, X^0)$$

$$\int_0^T \frac{\partial X_{g_2}}{\partial X^0}\left(T - s, X_{g_1}(s, X^0)\right)(V_{g_1} - V_{g_2})\Big|_{X_{g_1}(s,X^0)} dS = X_{g_1}(T, X^0) - X_{g_2}(T, X^0)$$

Identity (Stefanov-U, 1998)

$$\int_0^T \frac{\partial X_{g_2}}{\partial X^0}\left(T-s, X_{g_1}(s,X^0)\right)(V_{g_1}-V_{g_2})\Big|_{X_{g_1}(s,X^0)} dS = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)$$

$$V_{g_j} := \left(\frac{\partial H_{g_j}}{\partial \xi}, -\frac{\partial H_{g_j}}{\partial x}\right)$$ the Hamiltonian vector field.

Particular case:

$$(g_k) = \frac{1}{c_k^2}\left(\delta_{ij}\right), \quad k = 1,2$$

$$V_{g_k} = \left(c_k^2 \xi,\ -\frac{1}{2}\nabla(c_k^2)|\xi|^2\right)$$

Linear in c_k^2!

Reconstruction

$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\Big(T-s, X_{g_2}(s,X^0)\Big) \times \Big((c_1^2-c_2^2)\xi,\ -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{g_2}(s,X^0)} dS = \underbrace{X_{g_1}(T,X^0)}_{\text{data}} - X_{g_2}(T,X^0)$$

Inversion of weighted geodesic ray transform and use similar methods to U-Vasy.

The Linear Problem: General Case

The linearization of the map $g \to d_g$ leads to the question of invertability of the integration of two tensors along geodesics.

Let $f = f_{ij}\, dx^i \otimes dx^j$ be a symmetric 2-tensor in M. Define $f(x,\xi) = f_{ij}(x)\xi^i\xi^j$. The *ray transform* of f is

$$I_2 f(x,\xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))\, dt, \quad x \in \partial M, \xi \in S_x M,$$

where φ_t is the geodesic flow,

$$\varphi_t(x,\xi) = (\gamma(t,x,\xi), \dot\gamma(t,x,\xi)).$$

In coordinates

$$I_2 f(x,\xi) = \int_0^{\tau(x,\xi)} f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\, dt.$$

The Linear Problem: General Case

Recall the Helmholtz decomposition of $F : \mathbb{R}^n \to \mathbb{R}^n$,

$$F = F^s + \nabla h, \quad \nabla \cdot F^s = 0.$$

Any symmetric 2-tensor f admits a *solenoidal decomposition*

$$f = f^s + dh, \quad \delta f^s = 0, \quad h|_{\partial M} = 0$$

where h is a symmetric 1-tensor, $d = \sigma \nabla$ is the inner derivative (σ is symmetrization), and $\delta = d^*$ is divergence.

By the fundamental theorem of calculus, $I_2(dh) = 0$ if $h|_{\partial M} = 0$. I_2 is said to be *s-injective* if it is injective on solenoidal tensors.

Local Result for Linearized Problem

Theorem (Stefanov-U-Vasy, 2014). Let f be a symmetric tensor field of order 2. let $p \in \partial M$ be a strictly convex point. Assume that $I_2(f)(\gamma) = 0$ for all geodesics γ joining points near p. Then f is s-injective near p.

This is a Helgason type support theorem for tensor fields of order 2. The only previous result was for real-analytic metrics (Krishnan).

After this one uses pseudolinearization again to obtain the local boundary rigidity result.

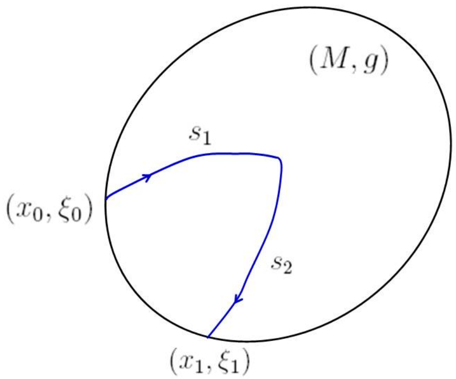
A global boundary rigidity result is expected to be obtained in the same way as the isotropic case assuming the foliation condition.

REFLECTION TRAVELTIME TOMOGRAPHY

Broken Scattering Relation

(M, g): manifold with boundary with Riemannian metric $\boxed{g}$

$$\boxed{\begin{gathered}((x_0, \xi_0), (x_1, \xi_1), t) \in \mathcal{B} \\ t = s_1 + s_2\end{gathered}}$$



Theorem (Kurylev-Lassas-U)

$n \geq 3$. Then ∂M and the broken scattering relation $\mathcal{B}$ determines (M, g) uniquely.

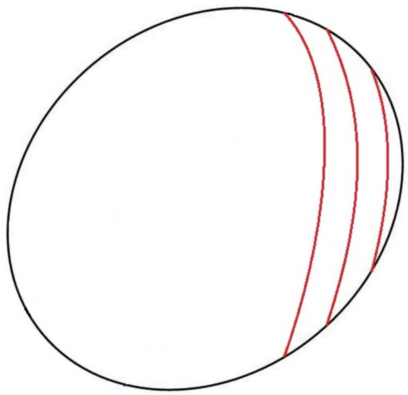
Numerical Method

(Chung-Qian-Zhao-U, IP 2011)

$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\Big(T-s, X_{g_2}(s,X^0)\Big) \times \Big((c_1^2-c_2^2)\xi,\ -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{g_2}(s,X^0)} dS = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)$$

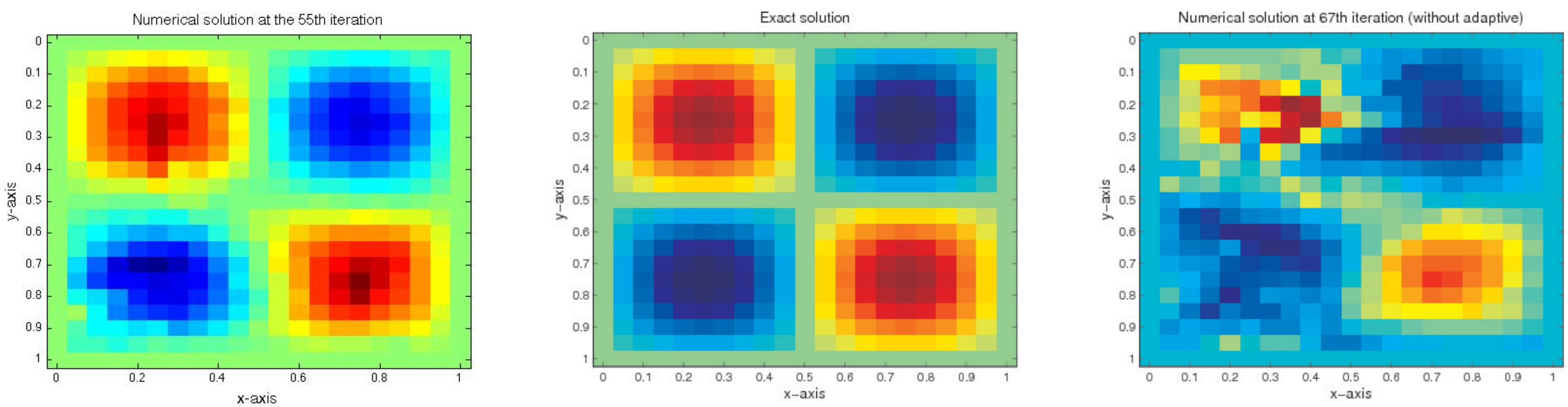
Adaptive method

Start near $\partial\Omega$ with $c_2 = 1$ and iterate.



Numerical examples

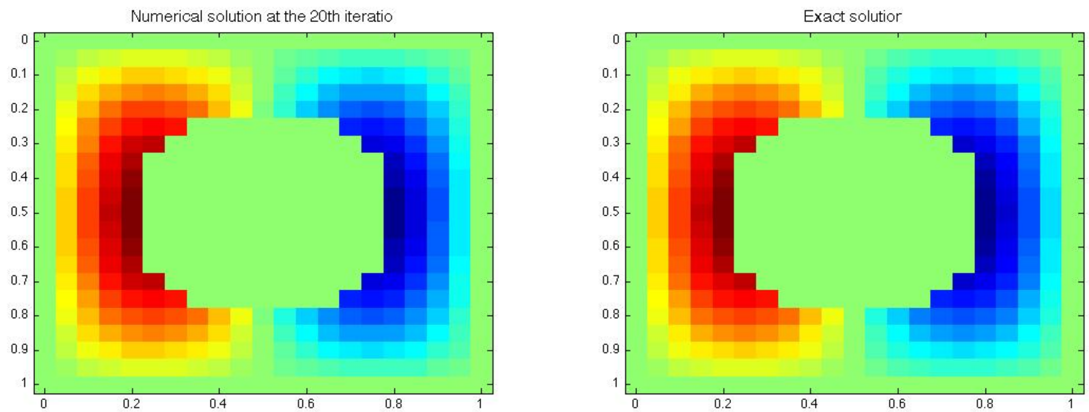
Example 1: An example with no broken geodesics, $c(x,y) = 1 + 0.3\sin(2\pi x)\sin(2\pi y)$, $c_0 = 0.8$.



Left: Numerical solution (using adaptive) at the 55-th iteration. Middle: Exact solution. Right: Numerical solution (without adaptive) at the 67-th iteration.

Example 2: A known circular obstacle enclosed by a square domain. Geodesic either does not hit the inclusion or hits the inclusion (broken) once.

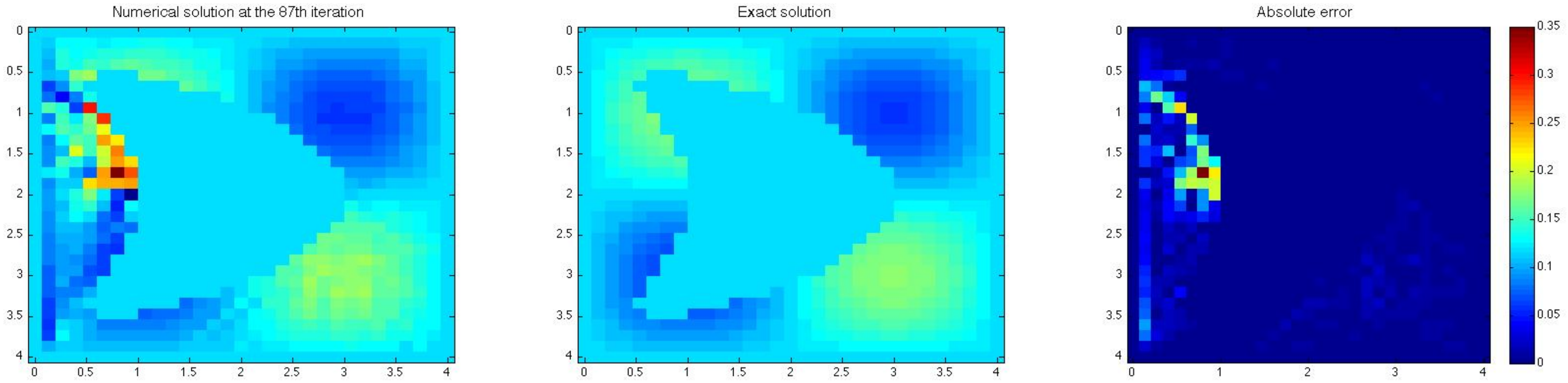
$c(x, y) = 1 + 0.2 \sin(2\pi x) \sin(\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 20-th iteration. The relative error is 0.094%. Right: Exact solution.

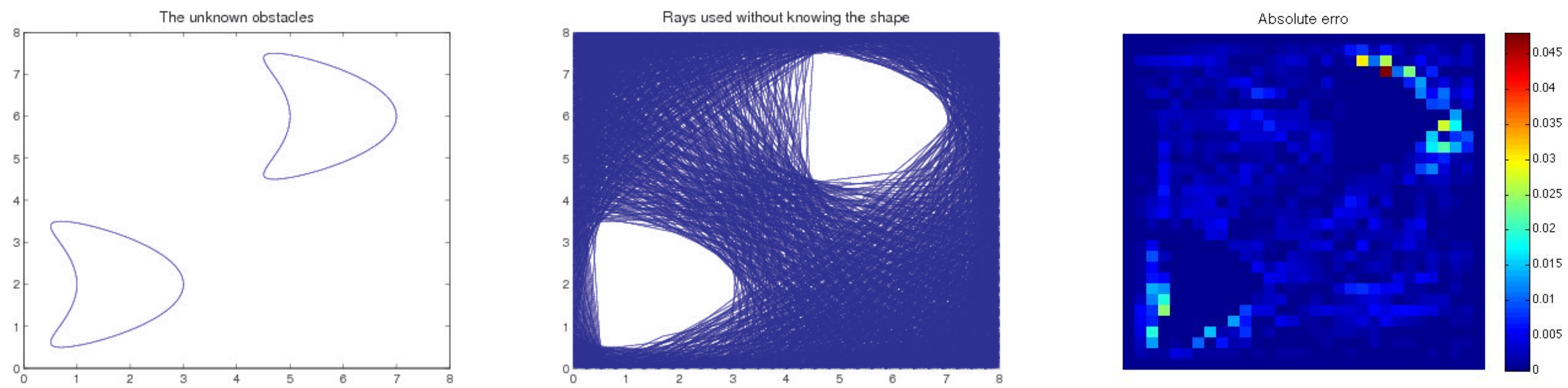
Example 3: A concave obstacle (known).

$c(x, y) = 1 + 0.1 \sin(0.5\pi x) \sin(0.5\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 117-th iteration. The relative error is 2.8%. Middle: Exact solution. Right: Absolute error.

Example 4: Unknown obstacles and medium.

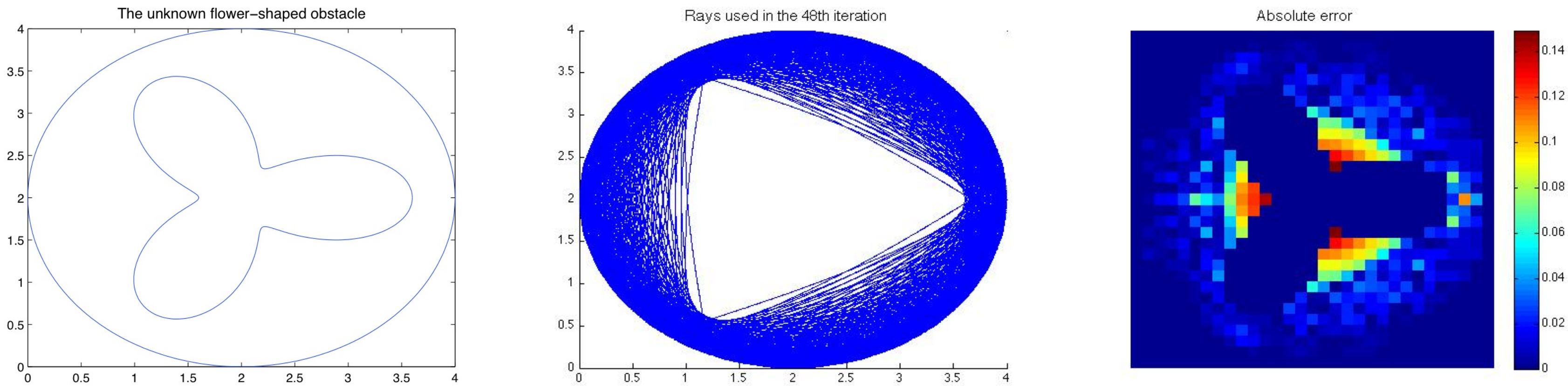


Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

Example 4: Unknown obstacles and medium (continues).

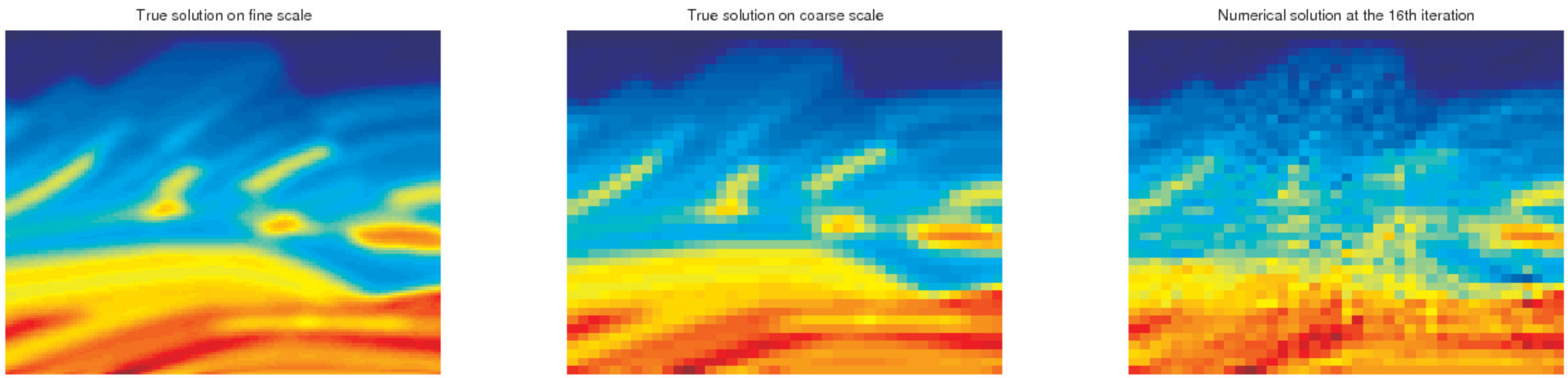
$r = 1 + 0.6\cos(3\theta)$ with $r = \sqrt{(x-2)^2 + (y-2)^2}$.

$$c(r) = 1 + 0.2\sin r$$



Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

Example 5: The Marmousi model.



Left: The exact solution on fine grid. Middle: The exact solution projected on a coarse grid. Right: The numerical solution at the 16-th iteration. The relative error is 2.24%.