Inverse scattering and Calderón's problem. Tools: a priori estimates

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- Solutions of $(\Delta + k^2)u = 0$:
  Trace Theorems.
  Restriction of the Fourier transform.
  Herglotz wave functions.
- The free resolvent. Uniform Sobolev estimates.
- Consequences. Further results.

The model case: The homogeneous free equation

The non uniqueness of the $\mathbb{R}^n$-problem

$$(\Delta + k^2)u = f$$

is due to the existence of entire solutions of the homogeneous equation

$$(\Delta + k^2)u = 0$$

the so called generalized eigenfunction, this fact makes Helmholtz equation of hyperbolic type, its Fourier symbol vanishes on the sphere of radius k. A class of solutions of the homogeneous equation are the plane waves parameterized by its frequency k and its direction ω (the direction of its wave front set).

$$\psi_0(k, \omega, x) = e^{ik\omega \cdot x} \tag{1}$$

The Fourier transform of this function is a Dirac delta at the point $k\omega$ on the sphere of radius k.

Herglotz wave functions

In scattering theory an important role is played by the superposition with a density $g(\omega)$ of plane waves, namely

$$u_i(x) = \int_{S^{n-1}} e^{ik\omega\cdot x} g(\omega) d\sigma(\omega). \qquad (2)$$

If g is a function in $L^2(S^{n-1})$, u_i is called a Herglotz wave function, which is also an entire solution of the homogeneous Helmholtz equation.
For the solvability of inverse problems, the Herglotz wave functions are important, see [CK]. For instance the scattering amplitudes used in inverse problems (either in the acoustic, the Schrödinger or the obstacle inverse problems) are not dense in $L^2(S^{n-1})$, if there exists a solution of an associated problem which is a Herglotz wave function. This density property is cruptial from the spectral point of view.

F.T. of measures

Herglotz wave functions are just the distributional Fourier transforms of $L^2(S^{n-1})$-densities on the sphere. They are in the range of the operator "extension of the Fourier transform"

$$E_k(g)(x) = \widehat{gd\sigma}(kx), \tag{3}$$

for a function $g \in L^2(S^{n-1})$.

Theorem (Herglotz, Hartman and Wilcox)

An entire solution v of the equation $(\Delta + k^2)v = 0$ is a H. w. f. if and only if it satisfies

$$\sup_R \frac{1}{R} \int_{|x|<R} |v(x)|^2 dx < \infty. \tag{4}$$

Furthermore if g is its density, we have

$$\limsup_{R\to\infty} \frac{1}{R} \int_{|x|<R} |v(x)|^2 dx \sim Ck^{n-1} \|g\|^2_{L^2(S^{n-1})} \tag{5}$$

F.T. of measures

We give a geometric proof, extended to the Fourier transform of measures carried on submanifolds of codimension d in $\mathbf{R}^n$ and whose density is in L^2.
Consider the case $k = 1$. If u is a tempered distribution solution of $(\Delta + 1)u = 0$, then its Fourier transform $\hat{u}$ is supported in S^{n-1}, denoting $g = \hat{u}$, we can rewrite the first statement of above theorem as a special case of

Theorem (Agmon-Hörmander)

Let M be a $\mathcal{C}^1$ submanifold of codimension d in $\mathbf{R}^n$. Let us denote by $d\sigma$ the induced measure. Assume that K is a compact subset of M. If $u \in \mathcal{S}'$ with Fourier transform supported in K and given by an $L^2(M)$-function, $\hat{u} = g(\xi)d\sigma(\xi)$ then there exists $C > 0$

$$\sup_{R>0} \frac{1}{R^d} \int_{|x|\leq R} |u(x)|^2 dx \leq C \int_M |g(\xi)|^2 d\sigma(\xi). \qquad (6)$$

Proof

By using a partition of unity we may assume that K is small and we can describe M by the equation $\xi'' = h(\xi')$, where $\xi' = (\xi_1, ..., \xi_{n-d})$ and $\xi'' = (\xi_{n-d+1}, ..., \xi_n)$ and $h \in \mathcal{C}^1$.
Let us write the measure $d\sigma = a(\xi')d\xi'$, for a positive and continuous function a, we have
$\hat{u}(\xi) = \hat{u}(\xi', h(\xi'))d\sigma = g(\xi')a(\xi')d\xi'$ and

$$u(x) = \hat{u}(e^{ix\cdot\xi}) = (2\pi)^{-n}\int_{\mathbf{R}^{n-d}} e^{i(x'\cdot\xi' + x''\cdot h(\xi'))}g(\xi')a(\xi')d\xi'$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^{n-d}} e^{ix'\cdot\xi'}F(x'', \xi')d\xi',$$

where $F(x'', \xi') = e^{ix''\cdot h(\xi')}g(\xi')a(\xi')$. By Plancherel formula in x' we have

$$\int_{\mathbf{R}^{n-d}} |u(x', x'')|^2 dx' = \int_{\mathbf{R}^{n-d}} |\widehat{u(\cdot, x'')}(\xi')|^2 d\xi'$$

$$= \int_{\mathbf{R}^{n-d}} |F(x'', \xi')|^2 d\xi' = \int_{\mathbf{R}^{n-d}} |\hat{g}(\xi')|^2 a(\xi')^2 d\xi' \leq C\|g\|^2_{L^2(M)}.$$

$$\frac{1}{R^d}\int_{B_R} |u(x)|^2 dx' dx'' \leq$$

$$\frac{1}{R^d}\int_{[-R,R]^d}\int_{\mathbf{R}^{n-d}} |u(x', x'')|^2 dx' dx'' \leq C\|g\|^2_{L^2(M)}$$

□

Corollary

Assume $g \in L^2(S^{n-1})$ and let us define

$$u(x) = \int_{S^{n-1}} e^{ik\theta\cdot x} g(\theta) d\sigma(\theta).$$

Then

$$|||v|||_*^2 := \sup_{R\geq 0} \frac{1}{R}\int_{|x|<R} |u(x)|^2 dx \leq Ck^{n-1}\|g\|^2_{L^2(S^{n-1})}. \tag{7}$$

Equivalence

Theorem

Assume $u \in L^2_{loc} \cap \mathcal{S}'$ *such that*

$$\limsup_{R\to\infty} \frac{1}{R^d} \int_{|x|<R} |u(x)|^2 dx < \infty.$$

Let Ω *be an open set in* $\mathbf{R}^n$ *such that* $\hat{u}$ *restricted to* Ω, $g = \hat{u}_{|\Omega}$, *is compactly supported in a* $\mathcal{C}^\infty$*-submanifold* M *of codimension* d, *then* $g \in L^2(M)$, *and furthermore*

$$\int_M |g|^2 d\sigma \leq C \limsup_{R\to\infty} \frac{1}{R^d} \int_{|x|<R} |u(x)|^2 dx. \tag{8}$$

The mollification

Lemma

Let $u \in L^2_{loc} \cap \mathcal{S}'$ and $\chi \in \mathcal{C}_0^\infty$ supported on $B(0,1)$, and denote $g_\epsilon = \hat{u}(\cdot) \star \epsilon^{-n}\chi(\cdot/\epsilon)$. Then, for fixed $d > 0$, we have

$$\|g_\epsilon\|_{L^2}^2 \leq C_d(\chi)\epsilon^{-d}K_d(\epsilon),$$

where $K_d(\epsilon) = \sup_{R\epsilon \geq 1} \frac{1}{R^d} \int_{|x|<R} |u(x)|^2 dx$ and $C_d(\chi)$ only depends on χ.

$$\|g_\epsilon\|_{L^2}^2 = \|u(\cdot)\hat{\chi}(\epsilon(\cdot))\|_{L^2}^2 = \Big(\int_{|\epsilon x|\leq 1} + \sum_{j=1}^{\infty} \int_{2^{j-1}\leq|\epsilon x|\leq 2^j}\Big)|u(x)\widehat{\chi}(\epsilon x)|^2 dx$$

$$\leq \sup_{|y|\leq 1} \hat{\chi}(y)^2 \int_{|\epsilon x|\leq 1} |u(x)|^2 + \sum_{j=1}^{\infty} \sup_{2^{j-1}\leq|y|\leq 2^j} \widehat{\chi}(y)^2 \int_{2^{j-1}\leq|\epsilon x|\leq 2^j} |u(x)|^2$$

$$\leq \epsilon^{-d}\Big(\sup_{|y|\leq 1} |\widehat{\chi}(y)|^2 \epsilon^d \int_{|\epsilon x|\leq 1} |u(x)|^2 dx$$

$$+ \sum_{j=1}^{\infty} \sup_{2^{j-1}\leq |y|\leq 2^j} |\widehat{\chi}(y)|^2 2^{jd} \cdot \sup_{j=1,2,\dots} (\frac{2^j}{\epsilon})^{-d} \int_{2^{j-1}\leq |\epsilon x|\leq 2^j} |u(x)|^2 dx\Big)$$

$$\leq \epsilon^{-d} C_d(\chi) \sup_{\epsilon R\geq 1} R^{-d} \int_{B(0,R)} |u(x)|^2 dx.$$

where

$$C_d(\chi) = \sup_{|y|\leq 1} |\widehat{\chi}(y)|^2 + \sum_{j=1}^{\infty} \sup_{2^{j-1}\leq |y|\leq 2^j} |\widehat{\chi}(y)|^2 2^{jd}.$$

Going back to the proof of the theorem, let us see that g is an L^2-density on M.

End of equivalence

Since g is supported on M, then g_ϵ is supported on $M_\epsilon = \{x \in \mathbf{R}^n : d(x,M) \leq \epsilon\}\}$; since $u \in \mathcal{S}'$ then $g_\epsilon \to g$ en $\mathcal{S}'$. Let us take a test functions $\psi \in \mathcal{C}_0^\infty$:

$$|g(\psi)| = |\lim_{\epsilon\to 0}(g_\epsilon)(\psi)| = \lim_{\epsilon\to 0} |\int_{M_\epsilon} (\hat{u} \star \chi_\epsilon(x))\psi(x)dx|$$

$$\leq \lim_{\epsilon\to 0}(\int |g_\epsilon|^2 dx)^{1/2}(\int_{M_\epsilon} |\psi(x)|^2 dx)^{1/2}$$

$$\leq \lim_{\epsilon\to 0}(\epsilon^{-d}\int_{M_\epsilon} |\psi(x)|^2 dx)^{1/2}(K_d(\epsilon)C_d)^{1/2},$$

and hence, since $\epsilon^{-d}\int_{M_\epsilon} |\psi(x)|^2 dx \to \int_M |\psi(x)|^2 d\sigma(x)$, we have

$$|g(\psi)| \leq \limsup_{\epsilon\to 0} K(\epsilon)^{1/2}\|\psi\|_{L^2(M)} C_d^{1/2},$$

this means that g is a function in $L^2(M)$ such that

$$\int_M |g(\theta)|^2 d\sigma(\theta) \leq C_d \limsup_{\epsilon\to 0} K(\epsilon),$$

Corollary

A solution of the homogeneous Helmholtz equation is a Herglotz wave function if and only if Hartman-Wilcox condition (4) holds

Notice that the theorems give us an equivalence of $L^2(S^{n-1})$-norm of the density with the norm $|||\cdot|||_*$ of the solution.
Let us define the Besov space

$$B_s = \{v \in L^2_{loc} : \|v\|_{B_s} = \sum_{j=0}^{\infty} R^s_{j+1} (\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty\}, \qquad (9)$$

where $R_j = 2^{j-1}$ if $j \geq 1$, $R_0 = 0$ and $\Omega_j = \{x : R_j \leq |x| \leq R_{j+1}\}$.
The elements of the dual B^*_s are the functions $v \in L^2_{loc}$ with

$$\|v\|^2_{B^*_s} = \sup_{j=1,2\ldots} R_j^{-2s} \int_{\Omega_j} |v|^2 dx \leq \infty \qquad (10)$$

This norm is equivalent to $|||\cdot|||_*$ with $2s = d$, when the supremum there is taken over $R > 1$.

Corollary

Let M be a $\mathcal{C}^1$-submanifold in $\mathbf{R}^n$ of codimension d and K a compact contained in M. Then the operator given by the restriction of the Fourier transform to K, defined for $v \in \mathcal{S}$ as

$$T(v) = \hat{v}_{|K} \in L^2_K(d\sigma) \tag{11}$$

can be extended by continuity to an onto map from $B_{d/2}$ to $L^2_K(d\sigma)$.

The adjoint of T, defined for $\psi \in L^2_K(d\sigma)$ by

$$T^*(\psi) = \widehat{(\psi d\sigma)} \in B^*_{d/2} \tag{12}$$

is one to one from (8) and has closed range, hence T is onto, see [Rudin,Thm 4.15.]

This corollary is a dual trace theorem at the end point, which means that it gives a substitute of the Sobolev space $W^{d/2,2}(\mathbf{R}^n)$ in order to obtain traces in L^2 when restricted to a submanifold of codimension d. To compare with, let us recall the classical trace theorem in Sobolev spaces.

Theorem

Let M be a $\mathcal{C}^\infty$ manifold of codimension d and $\alpha > d/2$, then there exists a bounded operator

$$\tau : W^{\alpha,2}(\mathbf{R}^n) \to W^{\alpha-d/2,2}(M),$$

such that for $\psi \in \mathcal{C}_0^\infty$, $\tau(\psi) = \psi_{|_M}$. This operator is called the trace operator on M and $\tau(f)$ the trace of f on M which we also denote by $f_{|M}$.

Dual trace theorem

Given $\epsilon \geq 0$, write $\alpha = d/2 + \epsilon$. Then for every $\epsilon \geq 0$,

$$\|\tau(g)\|_{L^2(M)} \leq C\|g\|_{W^{d/2+\epsilon,2}(R^n)} = \|\hat{g}\|_{L^2((1+|\xi|^2)^{d/2+\epsilon}d\xi)}.$$

If we take $\hat{g} = f$, we obtain that

$$\|\tau(\hat{f})\|_{L^2(M)} \leq C\|f\|_{L^2((1+|x|^2)^{d/2+\epsilon}dx)}.$$

This means that the restriction operator

$$Tf = \hat{f}_{|M} : L^2(1+|x|^2)^{d/2+\epsilon} \to L^2(M) \tag{13}$$

and its adjoint

$$T^*(\psi) = \widehat{(\psi d\sigma)} \tag{14}$$

$$\|T^*(\psi)\|_{L^2((1+|\xi|^2)^{-d/2-\epsilon}d\xi)} \leq C\|\psi\|_{L^2(M)}. \tag{15}$$

This inequality can also be written as

$$\sup_{R\geq 1} \frac{1}{R^{d/2+\epsilon}} \int_{B(0,R)} |T^*(\psi)(\xi)|^2 d\xi \leq C\|\psi\|^2_{L^2(M)}. \tag{16}$$

Estimate (6) is the above inequality for $\epsilon = 0$.

Stein-Tomas operator

Theorem

$$T^*T(f) = \widehat{d\sigma} \star f : B_{d/2} \to B^*_{d/2}. \quad (17)$$

In the case $M = \mathbf{S}^{n-1}$ this operator is related to the imaginary part of the free resolvent. Formula (17) gives a factorization of this imaginary part when considered as an operator from some space to its dual, by inserting the intermediate $L^2(\mathbf{S}^{n-1})$. Let us define

$$I_k f(x) = \frac{1}{k}(\widehat{d\sigma_k} * f)(x), \quad (18)$$

where $d\sigma_k$ is the measure on the sphere of radius k. Consider norm $|||\cdot|||_*$ and which is the dual of

$$\|u\|_{\tilde{B}_{1/2}} = \sum_{-\infty}^{\infty} \left(R_j \int_{\Omega_j} |u(x)|^2 dx \right)^{1/2}, \quad (19)$$

It allows dilations as opposite to $sup_{R>1}$ [Kenig, Ponce and Vega]

Corollary

There exists a constant $C > 0$ uniform in k such that

$$|||I_k f|||_* \leq Ck^{-1} \|f\|_{\tilde{B}_{1/2}} \tag{20}$$

Proof: We reduce to the case $k = 1$, noticing that for $u_k(x) = u(x/k)$, we have

$$|||f_k|||_* = k^{(n-1)/2} |||f|||_*,$$

$$\|f_k\|_{\tilde{B}_{1/2}} = k^{(n+1)/2} \|f\|_{\tilde{B}_{1/2}}$$

and

$$(I_k f)(x/k) = k^{-2} I_1(f_k)(x).$$

The case $k = 1$ can be proved from Hartman-Wilcox, duality and T^*T-argument.

The Restriction of the Fourier Transform

Given $f \in L^p(\mathbf{R}^n)$ and a submanifold M in $\mathbf{R}^n$, when does it make sense to restrict $\hat{f}$ to M in the sense of this restriction being a function in $L^t(M)$? We are going to study the case $M = \mathbf{S}^{n-1}$, starting with $t = 2$. It is important to remark that in these theorems the positiveness of curvature of the sphere plays a fundamental role. We start will the dual theorem

Theorem (Extension theorem)

Let ψ be an $L^2(\mathbf{S}^{n-1})$ density, then its extension $T^*\psi = \widehat{\psi d\sigma}$ is in $L^q(\mathbf{R}^n)$, for $q \geq \frac{2(n+1)}{n-1}$, i.e. if q satisfies the relation

$$1/2 - 1/q \geq 1/(n+1). \tag{21}$$

Furthermore we have the estimate

$$\|T^*\psi\|_{L^q(\mathbf{R}^n)} \leq C\|\psi\|_{L^2(S^{n-1})}. \tag{22}$$

The Restriction theorem

Corollary (Stein-Tomas)

Let $f \in L^p(\mathbf{R}^n)$ where $p \leq \frac{2(n+1)}{n+3}$ we can define $Tf = \hat{f}_{|_{S^{n-1}}}$ as a L^2 density and it holds

$$\|Tf\|_{L^2(\mathbf{S}^{n-1})} \leq C\|f\|_{L^p} \qquad (23)$$

Remark 1: The range of q is sharp: Take a non negative function $\phi \in \mathcal{C}_0^\infty$ and construct $(\phi_\delta)(\xi', \xi_n) = \phi(\frac{\xi'}{\delta}, \frac{\xi_n - e_n}{\delta^2})$. Then

$$\widehat{\phi_\delta}(x) = e^{i\delta^2 x_n}\delta^{n+1}\phi(\delta x', \delta^2 x_n)$$

and hence $\|T\widehat{\phi_\delta}\|_{L^t(S^{n-1})} = \|\phi_\delta\|_{L^t(S^{n-1})} \geq C\delta^{(n-1)/t}$ and $\|\widehat{\phi_\delta}\|_{L^p} \leq C\delta^{n+1-(n+1)/p}$, where $p' = q$. Assume that

$$\|T\psi\|_{L^t(\mathbf{S}^{n-1})} \leq C\|\psi\|_{L^p}, \qquad (24)$$

and take $\psi = \widehat{\phi_\delta}$, then if $\delta \to 0$, we obtain the necessary condition $(n+1)/q = (n+1)(1 - \frac{1}{p}) < (n-1)/t$.

Remark 2: There is another necessary condition that comes from the evaluation of $T^*(1)$ in term of Bessel function given by Funk-Ecke formula. That is the constrain $q > 2n/(n-1)$. The sufficiency of $(n+1)/q < (n-1)/t$, together with $q > 2n/(n-1)$ to have inequality (24) is known as the "Restriction conjecture", an open question in classical Fourier Analysis. Notice that in the particular case $t = 2$ we obtain the range of the Corollary.
Remark 3: We can write for ω_n the measure of the sphere:

$$|\widehat{\psi d\sigma}(\xi)| = \int_{\mathbf{S}^{n-1}} e^{ix\cdot\xi}\psi(x)d\sigma(x)| \leq \|\psi\|^{1/2}_{L^2(S^{n-1})}\omega_n^{1/2}.$$

Then it suffices to prove the theorem at the end point $q = \frac{2(n+1)}{n-1}$.

Estimate of the mollification with resolution ϵ of a measure on the sphere with density f.

Lemma

Let $\chi \in \mathcal{S}$, denote $d\sigma_\epsilon = f(\cdot)d\sigma(\cdot) \star \epsilon^{-n}\chi(\cdot/\epsilon)$, where $f \in L^\infty(\mathbf{S}^{n-1})$ then

$$sup_x|d\sigma_\epsilon(x)| \leq C\epsilon^{-1}$$

Proof: We make a reduction to the case where χ is compactly supported("Schwartz tails argument)": Take a $\mathcal{C}_0^\infty$ partition of unity in $\mathbf{R}^n$ such that

$$\sum_{j=0}^{\infty} \psi_j(x) = 1,$$

where ψ_0 is supported in $B(0,1)$ and $\psi_j = \psi(2^{-j}x)$ for $j > 0$, and ψ is supported in $1/2 \leq |x| \leq 2$.

We take

$$\sum_{j=0}^{\infty} \psi_j(\epsilon^{-1}x) = 1.$$

Now write

$$d\sigma_\epsilon(x) = d\sigma(\cdot) \star \epsilon^{-n} \sum \psi_j(\epsilon^{-1}(\cdot))\chi(\cdot/\epsilon),$$

Notice that the jth term , $j > 0$ is an integral on the sphere of radius 1 centered at x of a function supported on the annulus $2^{j-1}\epsilon \leq |y| \leq 2^j\epsilon$. In this annulus, since χ is rapidly decreasing, we have:

$$|\chi(\epsilon^{-1}x)| \leq \frac{C_N}{(1+2^j)^N},$$

Hence

$$|d\sigma(\cdot) \star \epsilon^{-n}\psi_j(\epsilon^{-1}(\cdot))\chi(\cdot/\epsilon)(x)| \leq C_N \frac{(2^j\epsilon)^{n-1}}{(1+2^j)^N}\epsilon^{-n}.$$

By taking N big enough, the sum in j converges bounded by $C\epsilon^{-1}$. (The term $j = 0$, satisfies trivially the inequality) .

Proof of the extension thm.

Case the case $L^2 \to L^{p'}$, for $p' > \frac{2(n+1)}{n-1}$, i.e.
$1/2 - 1/p' > 1/(n+1)$.
T^*T-argument:
Stein-Tomas operator: $f \to \widehat{d\sigma} * f = K(f)$ bounded from
$L^p \to L^{p'}$:
Facts $\widehat{d\sigma}(x) = C_n \frac{J_{\frac{n-2}{2}}(|x|)}{|x|^{\frac{n-2}{2}}}$
Asymptotics and dyadic decomposition.
$K_j(f) = \psi_j \widehat{d\sigma} * f$
Interpolation $\|K_j\|_{L^1 \to L^\infty} \leq C2^{-j(n-1)/2}$ and $\|K_j\|_{L^2 \to L^2} \leq C2^j$
Geometric series.

It follows by dilations:

Corollary

If v is a Herglotz wave function corresponding to the eigenvalue k^2 with density g, then for

$$1/2 - 1/q \geq 1/(n+1)$$

it holds

$$\|v\|_{L^q} \leq Ck^{-n/q}\|g\|_{L^2(S^{n-1})}. \tag{25}$$

Corollary

Let $k > 0$, and consider $I_k f(x) = \frac{1}{k}(\widehat{d\sigma_k} * f)(x)$. Then, for

$$\frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1} \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

we have $\|I_k f\|_{L^q} \leq Ck^{n(\frac{1}{p}-\frac{1}{q})-2}\|f\|_{L^p}$

Transmision eigenvalues

Estimates for the free resolvent

The outgoing solution of the equation in $\mathbf{R}^n$

$$(\Delta + k^2)u = f \tag{26}$$

is the function

$$u(x) = \int \Phi(x-y)f(y)dy.$$

In the F.T. side the outgoing fundamental solution $\Phi(x)$ is

$$\hat{\Phi}(\xi) = (-|\xi|^2 + k^2 + i0)^{-1}, \tag{27}$$

In terms of the homogeneous distributions of degree -1, We can obtain the expression from the one variable formula

$$\lim_{\epsilon \to 0+} (t + i\epsilon)^{-1} = pv\frac{1}{t} + i\pi\delta,$$

extended to the $\mathbf{R}^n$-function $t = H(\xi)$ as far as we can take locally H as a coordinate function in a local patch of a neighborhood in $\mathbf{R}^n$ at any point ξ_0 for which $H(\xi_0) = 0$.

Proposition

Let $H : \mathbf{R}^n \to \mathbf{R}$ such that $|\bigtriangledown H(\xi)| \neq 0$ at any point where $H(\xi) = 0$, then we can take the distribution limit

$$(H(\xi) + i0)^{-1} = \lim_{\epsilon \to 0+} (H(\xi) + i\epsilon)^{-1}. \tag{28}$$

$$(H(\xi) + i0)^{-1} = pv \frac{1}{H(\xi)} + i\pi\delta(H) \tag{29}$$

in the sense of the tempered distributions.

The distribution $\delta(H)$ is defined as

$$\delta(H)(\psi) = \int_{H(\xi)=0} \psi(\xi)\omega(\xi),$$

where ω is any $(n-1)$-form such that $\omega \wedge dH = d\xi$. It is easily seen, from the change of variable formula, that this integral does not depends on the choice of the form ω. The existence of such ω can be proved by using local coordinates in $\mathbf{R}^n$ adapted to the manifold $H(\xi) = 0$.

Limiting absorption principle

Let α any function which does not vanish at the points ξ with $H(\xi) = 0$, then

$$\delta(\alpha H) = \alpha^{-1}\delta(H).$$

We can choose an orthonormal moving frame on the tangent plane to $H(\xi) = 0$, namely $\omega_1, ..., \omega_{n-1}$, for this frame we have $\omega_1 \wedge ... \wedge \omega_{n-1} \wedge \frac{dH}{|\bigtriangledown H|} = d\xi$, it follows that $\delta(|\bigtriangledown H|^{-1}H)$ is the measure $d\sigma$ induced by $\mathbf{R}^n$ on the hypersurface $H(\xi) = 0$ and hence

$$\delta(H) = |\bigtriangledown H|^{-1} d\sigma.$$

Let $H(\xi) = -|\xi|^2 + k^2$, then

Lemma

$$(H(\xi) + i0)^{-1} = \lim_{s \downarrow 0}(-|\xi|^2 + k^2 + is)^{-1} = pv\frac{1}{H(\xi)} + \frac{i\pi}{2k} d\sigma \quad (30)$$

The resolvent

$$R_+(k^2)(f)(x) = (\Delta + k^2 + i0)^{-1}(f)(x)$$
$$= p.v. \int_{\mathbf{R}^n} e^{ix\cdot\xi} \frac{\hat{f}(\xi)}{-|\xi|^2 + k^2} d\xi + \frac{i\pi}{2k} \widehat{d\sigma} * f(x). \qquad (31)$$

Estimates are given by estimates of the model $I_k f = \frac{i\pi}{2k}\widehat{d\sigma} * f(x)$.
Retriction Thm for the F.T. $\rightarrow$ Selfdual L^p-estimate [KRS]

Theorem

For

$$\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1} \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

we have $\|R_+(k^2)(f)\|_{L^q} \leq C k^{n(\frac{1}{p}-\frac{1}{q})-2} \|f\|_{L^p}$

End point dual trace thm → Selfdual Besov estimate [AH-KPV]

Theorem

There exists a constant $C > 0$ uniform in k such that

$$\sup_R \frac{1}{R} \int_{B(0,R)} |R_+(k^2)(f)|^2 \leq Ck^{-1} \|f\|_{\tilde{B}_{1/2}} \tag{32}$$

Example: Agmon-Hormander estimate $W(x) = (1 + |x|^2)^{-1/2-\epsilon}$
The model is $I_k = T^* T$. Hopefully [RV]

Theorem

Let

$$\frac{1}{n} \geq \frac{1}{p} - \frac{1}{2} \geq \frac{1}{n+1}.$$

Then

$$\|R_+(k^2)(f)\|_{\tilde{B}^*_{1/2}} \leq k^{n(\frac{1}{p}-\frac{1}{2})-\frac{3}{2}} \|f\|_{L^p} \tag{33}$$

Selfdual L^p. Proof

The restriction $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q}$ needs to be added, since the Fourier multiplier $(-|\xi|^2 + k^2)^{-1}$ behaves as a Bessel potential of order 2 when $|\xi| \to \infty$.

Lemma (Mollification)

Let $\chi \in \mathcal{S}$, and

$$P_\epsilon(\xi) = pv \frac{1}{-|(\cdot)|^2 + 1} * \epsilon^{-n} \chi(\cdot/\epsilon)(\xi),$$

then

$$|P_\epsilon(\xi)| \leq C\epsilon^{-1}$$

$$P_\epsilon(\xi) = -pv \left(\int_{1-\epsilon \leq |\eta| \leq 1+\epsilon} + \int_{1-\epsilon > |\eta|} + \int_{|\eta| > 1+\epsilon} \right) \chi_\epsilon(\xi - \eta) \frac{1}{|\eta|^2 - 1} d\eta$$
$$= I_1 + I_2 + I_3.$$

Let us write

$$I_1 = \lim_{\delta\to 0}\int_{\delta\leq|1-|\eta||\leq\epsilon}\chi_\epsilon(\xi-\eta)\frac{1}{|\eta|^2-1}d\eta$$

$$= \lim_{\delta\to 0}\left(\int_{1-\epsilon}^{1-\delta}+\int_{1+\delta}^{1+\epsilon}\right)\int_{S^{n-1}}\chi_\epsilon(\xi-r\theta)\frac{1}{r^2-1}r^{n-1}d\sigma(\theta),$$

Changing $r=2-s$ in the second integral we obtain

$$I_1 = \lim_{\delta\to 0}\int_{1-\epsilon}^{1-\delta}F(r,\xi)(r-1)^{-1}dr,$$

where

$$F(r,\xi) = \int_{S^{n-1}}\chi_\epsilon(\xi-r\theta)\frac{r^{n-1}}{(r+1)}d\sigma(\theta) - \int_{S^{n-1}}\chi_\epsilon(\xi-(2-r)\theta)\frac{(2-r)^{n-1}}{(3-r)}d\sigma(\theta) \tag{34}$$

If we observe that $F(1,\xi)=0$, we may write by the mean value theorem

$$| \int_{1-\epsilon}^{1-\delta} F(r,\xi)(r-1)^{-1} dr| \leq \epsilon \sup_{1-\epsilon \leq r \leq 1} |\frac{\partial F}{\partial r}(r,\xi)|. \qquad (35)$$

The radial derivative of the first integral in the definition of F, (34), is given by

$$\frac{\partial}{\partial r}(\frac{r^{n-1}}{r+1}) \int_{S^{n-1}} \chi_\epsilon(\xi - r\theta) d\sigma(\theta) + \frac{r^{n-1}}{r+1} \int_{S^{n-1}} \theta \cdot \nabla \chi_\epsilon(\xi - r\theta) d\sigma(\theta).$$

The second of these integrals can be written as

$$\epsilon^{-1} \sum_{i=1}^{n} \frac{r^{n-1}}{r+1} \int_{S^{n-1}} \theta_i (\frac{\partial}{\partial x_i} \chi)_\epsilon (\xi - r\theta) d\sigma(\theta),$$

both integrals are mollifications with resolution ϵ of the measures $d\sigma(\theta)$ and $\theta_i d\sigma(\theta)$, which, from lemma 8, are bounded by $C(\chi)\epsilon^{-1}$. We have then $|\frac{\partial F}{\partial r}(r,\xi)| \leq C\epsilon^{-2}$, and hence $|I_1| \leq C\epsilon^{-1}$.

Proof of selfdual L^p-estimate

Theorem

[KRS] Let $a \in \mathbf{C}^n$ and $b \in \mathbf{C}$ such that $\Re b + |\Im a|^2/4 \neq 0$, then for any $u \in \mathcal{C}_0^\infty$, there exists C independent of a and b such that, for $1/p - 1/q \in [2/(n+1), 2/n]$

$$\|u\|_q \leq C|\Re b + |\Im a|^2/4|^{(1/p-1/q)n/2-1}\|(\Delta + a \cdot \nabla + b)u\|_p. \quad (36)$$

It contains the Carleman estimate and also Fadeev operator estimate and for $1/p - 1/q = 2/n$ uniform Sobolev.

Corollary

Let $\rho \in \mathbf{C}^n$ such that $\rho \cdot \rho = 0$. Assume that $\frac{2}{n} \geq \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ if $n > 2$ and $1 > \frac{1}{p} - \frac{1}{q} \geq \frac{2}{3}$ if $n = 2$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a constant C independent of ρ and f such that

$$\|f\|_{L^q} \leq C|\rho|^{n(\frac{1}{p}-\frac{1}{q})-2}\|(\Delta + \rho \cdot \nabla)f\|_{L^p} \quad (37)$$

Sketch of proof

1.

Theorem

Let $z \in \mathbf{C}$, p and q in the range of theorem and $u \in \mathcal{C}_0^\infty$ then there exists a constant C independent of z such that

$$\|u\|_q \leq C|z|^{(1/p-1/q)n/2-1}\|(\Delta + z)u\|_p \qquad (38)$$

Proof: Use Phragmen-Lindelöv maximum principle

Proposition

Let $F(z)$ analytic in the open half complex plane $\{Imz > 0\} = \mathbf{C}_+$ and continuous in the closure. Assume that $|F(z)| \leq L$ in $\partial\mathbf{C}_+$ and that for any $\epsilon > 0$ there exists C such that $|F(z)| \leq Ce^{\epsilon|z|}$ as $|z| \to \infty$ uniformly on the argument of z. Then $|F(z)| \leq L$ for any $z \in \mathbf{C}_+$.

u and v in a dense class

$$F(z) = z^{-(1/p-1/q)n/2+1} \int v(\Delta + z)^{-1} u$$

$$= z^{-(1/p-1/q)n/2+1} \int (-|\xi|^2 + z)^{-1} \hat{v}(\xi)\hat{u}(\xi) d\xi,$$

Continuity: Limiting absorption principle.
Boundedness at the boundary, estimates for resolvent
$F(z) \leq C\|u\|_p \|v\|_p$. □

General case: Reduce by phase shifts, rotations and dilations to prove: There exists $C_2 > 0$ such that for any real numbers ϵ and β and any $u \in \mathcal{C}_0^\infty$

$$\|u\|_q \leq C_2 \|(\Delta + \epsilon(\frac{\partial}{\partial y_1} + i\beta) \pm 1)u\|_p. \qquad (39)$$

Fourier multiplier

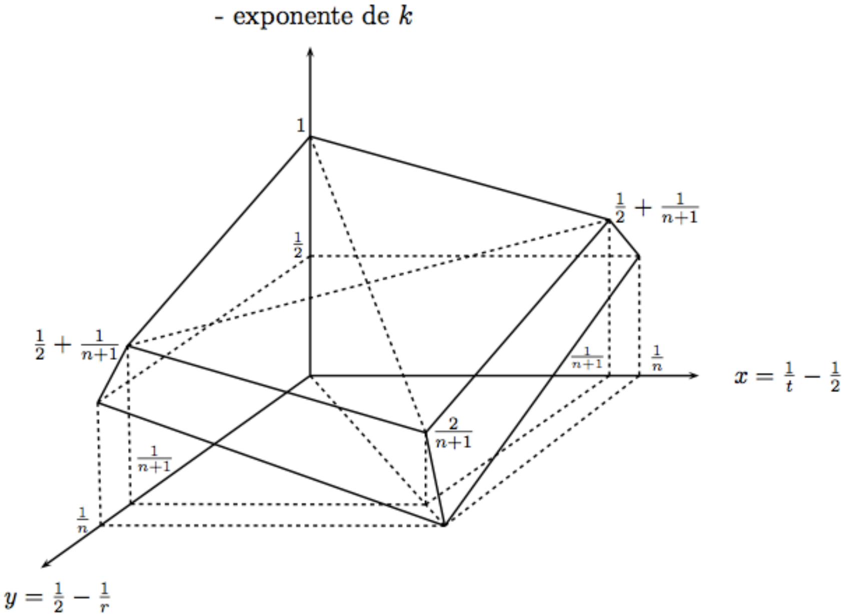
$$(Tf)(\xi) = m(\xi)\hat{f}(\xi),$$

where

$$m(\xi) = (-|\xi|^2 \pm 1 + i\epsilon(\xi_1 + \beta))^{-1}, \qquad (40)$$

End of proof

All estimates together



Extensions and open problems

1. Wave equation (Klein-Gordon) [KRS] useful in Control Theory.
2. Morrey-Campanato clases $Ł^{\alpha,p}$, [ChS], [CR], [W], [RV]: $p > 1$, $\alpha < n/p$.

$$\|V\|_{\alpha,p} = \sup_{x,R>0} R^{\alpha}(R^{-n}\int_{B(x,R)} |V(y)|^{p})^{1/p}$$

Case $\alpha = 2$, $p = n/\alpha$, $L^p = Ł^{\alpha,p}$. Uniform estimate

$$\|u\|_{L^2(V)} \leq C\|V\|^2_{\alpha,p}\|(\Delta + a\cdot\nabla + b)u\|_{L^2(V^{-1})}$$

Open range: $(\alpha = 2,\ 1 < p < (n-1)/2)$
Remark: L^p-selfdual estimate

$$\|u\|_{L^2(V)} \leq C\|V\|^2_{n/2}\|(\Delta + a\cdot\nabla + b)u\|_{L^2(V^{-1})}$$

Kato-Stummel Class.

3. Resolvent: X-rays transform class (open problem)

$$|||V|||_X = \sup_{x\in R^n,\omega\in S^{n-1}} \int_0^\infty V(x-t\omega)dt < \infty \qquad (41)$$

Radial case [BRV]

$$k\|R_+(k^2)f\|_{L^2(V)} + \|\nabla R_+(k^2)f\|_{L^2(V)} \leq C\|V\|_X^2\|f\|_{L^2(V^{-1})}$$

Stein conjecture.
4. Uniform estimates for lower order perturbations: extension of [AH] [KPV], Nirenberg-Walker estimate
5. In Riemannian manifolds ([DsfKS]) Resolvent ,Carleman

High energy reconstruction

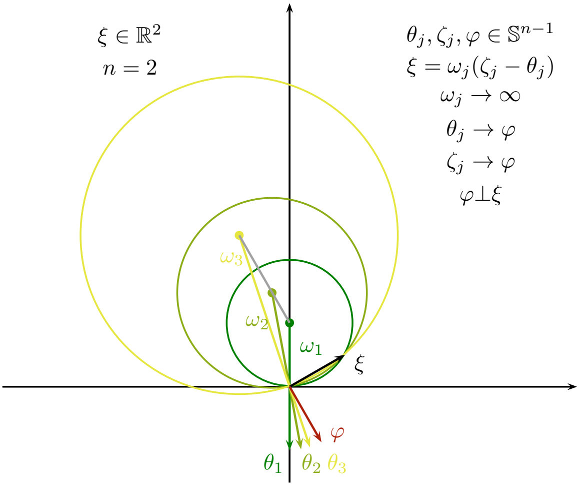


FIGURE 1. ξ belongs to spheres centered at $-\omega_j\theta_j$ with radii ω_j (Ewald spheres).