Inverse scattering and Calderón's problem

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Introduction. The Scattering Problem

Scattering is a part of the theory of perturbations. It trays to obtain information for a quantum hamiltonian H which is seen as a perturbation of an other H_0 which usually is called the "free hamiltonian".

We consider here the simplest case of the Schrödinger equation in $\mathbb{R}^n$ for an electrical potential $V(x)$, given by $H = \Delta + V(x)$ and the free one $H_0 = \Delta$.

We will study to basic problems that were the background of the powerful recent development of the field of Inverse Problems: The Gelfand-Fadeev Scattering problem and the Calderón electrical impedance tomography problem.

The point is to describe the perturbation, given by the potential V from external measurement to the object or by far field profiles of some special solutions that can be perturbations of solutions of the free system.

This measurements can be described by two operators :

1. The Dirichlet to Neumann operator on the boundary or
2. The scattering operator which can be obtained by comparison of the evolution of the two operators asymptotically in time (the existence of such operators is one of the problem in Spectral theory of operators).

In this presentation we will study the stationary point of view and later on we will comment the evolution approach.

Introduction. The Schrödinger equation

In the case of Schrödinger or potential scattering we assume that $V(x) = q(x) \in L^p$ for some p. The scattering solution of wave number k is the solution of the problem

$$\begin{cases} (\Delta + k^2)u = V(x)u \\ u = u_i + u_s \\ u_s \text{ satisfies the outgoing Sommerfeld radiation condition (S.R.C)} \end{cases} \tag{1}$$

The SRC is given by the asymptotic

$$\frac{d}{dr}u_s - iku_s = o(r^{-\frac{n-1}{2}}), \tag{2}$$

when $|x| = r \to \infty$.

u_i is an entire solution of the free hamiltonian, the homogeneous Helmholtz equation; this is the case of Herglotz wave functions which are superpositions of plane waves

$$u_i(x) = u_0(k, \theta, x) = e^{ik\theta\cdot x}, \tag{3}$$

The scattering solution $u = u(k, \theta, x)$ is a solution of the Helmholtz equation in the exterior of the support of V and so is the scattered solution u_s. Since it satisfies the S.R.C., it has the following asymptotics as $|x| \to \infty$:

$$u_s(x) = c_n k^{(n-3)/2} \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u_\infty(k, \theta, \frac{x}{|x|}) + o(|x|^{-(n-1)/2}) \quad (4)$$

The function $u_\infty(k, \theta, \omega)$, is known as the scattering amplitude or far field pattern, it represents the measurements in the inverse scattering problem: The influence of the perturbation in the free solution u_i. It is the response of the perturbation to the plane wave with energy k^2 and wave front direction θ.
The problem in inverse scattering deals with the recovery of the unknown potential V from the set of scattering data $u_\infty(k, \theta, \omega)$ for sets of energies k^2, incident directions $\theta \in \mathbb{S}^{n-1}$ and measurements directions $\omega \in \mathbb{S}^{n-1}$.

Program of the course

The problem will be approached by the following steps:

1. The direct problem. The unique continuation principle
2. Calderón Problem.
3. Estimates for the free resolvent and consequences.
4. Inverse scattering. Other problems

The direct problem

The way of constructing the scattering solutions is by mean of the so called Lippmann-Schwinger integral equation. We are going to state this equation, to sketch the proof of existence of the solution and to get an expression for the far field pattern which will be useful for our purposes.

Let us remark the equation satisfied by u_s,

$$(\Delta + k^2)u_s = Vu_i + Vu_s \tag{5}$$

The resolvent of the free problem gives the solution of

$$\begin{cases} (\Delta + k^2)u = f \\ u \text{ outgoing S.R.C} \end{cases} \tag{6}$$

The free resolvent

The solution is given by convolution with the outgoing fundamental solution,

$$\Phi_k(|x|) = \int_{R^n} \frac{e^{ikx\cdot\xi}}{-|\xi|^2 + k^2 + i0} d\xi.$$

This kernel has an explicit expression in term of Bessel-Hankel functions:

$$\Phi(x) = c_n k^{(n-2)/2} \frac{H^{(1)}_{(n-2)/2}(k|x|)}{|x|^{(n-2)/2}}, \text{ where } c_n = \frac{1}{2i(2\pi)^{(n-2)/2}}. \quad (7)$$

In the case of $n = 3$ this can be written as

$$\Phi(x) = \frac{e^{ik|x|}}{4\pi|x|}. \quad (8)$$

We call this the Resolvent operator $R_+ = R_+(k^2)$:

The direct problem

If we apply the resolvent to (5), since u_s is outgoing, we obtain

$$u_s = R_+(Vu).$$

This is the Lippmann-Schwinger integral equation

$$u_s(x) = \int_{\mathbf{R}^n} \Phi_k(x,y)V(y)(u_i(y) + u_s(y))dy \qquad (9)$$

Denote $< x >= (1 + |x|^2)^{1/2}$.

Theorem

Let $V \in L^r$, real, compactly supported with $r > n/2$, and $k > 0$. Then there exists a unique solution u_s of the Lippmann-Schwinger integral equation such that: $u_s \in W^{s,p'}(< x >^{-\beta} dx)$, where $s < 2 - n/r$ and $1/p - 1/p' = 1/r$, and β is some exponent depending on r such that $\beta < 1/2$ if $r < \infty$.

The direct problem. Existence and estimates

Proof: Take T_k the operator defined as $T_k(\psi) = R_+(V\psi)$. Write the Lippmann-Schwinger equation as

$$u_s = T_k(u_i + u_s) = R_+(Vu_i) + T_k(u_s)$$

We use Fredholm theorem.
To prove that T_k is compact in $W = W^{s,p'}(< x >^{-\beta} dx)$ we use the following estimate, which will be studied in the next session.

Theorem

Let $0 \leq 1/p - 1/p' = 1/r \leq 2/n$ if $n > 2$ or $0 \leq 1/p - 1/p' < 1$ for $n = 2$. Let $k > 0$ and $0 \leq s \leq 2 - n/r$. Then there exists a β depending on r and $\beta < 1/2$ if $r < \infty$, such that

$$\|R_+(k^2)f\|_{W^{s,p'}(<x>^{-\beta} dx)} \leq C(k)\|f\|_{L^p(<x>^{\beta} dx)} \tag{10}$$

The compactness of T_k follows from this theorem, Hölder inequality and Rellich compactness theorem.
Now to obtain the existence we need to prove that the only solution in $W^{s,p'}(< x >^{-\beta} dx)$ of the integral equation $u = T_k(u)$ is $u = 0$. To prove this, notice that u is a solution of the equation $(\Delta + k^2)u = Vu$ and since V is compactly supported we can use Rellich uniqueness theorem

Theorem

Let $D = \mathbb{R}^n \setminus \Omega$ be an exterior domain $\mathcal{C}^2$, assume that u is a $\mathcal{C}^2(D) \cup \mathcal{C}(\bar{D})$ outgoing solution of $(\Delta + k^2)u = 0$ such that

$$\Im \int_{\partial\Omega} u \frac{\partial \bar{u}}{\partial \nu} \geq 0, \tag{11}$$

where ν is the normal to D pointing out of D, then u vanishes in D

By the Green formula in a ball containing the support of V

$$\int_{\partial B} u \frac{\partial \bar{u}}{\partial \nu} = \int_B (\bigtriangledown u \cdot \bigtriangledown \bar{u} + V u \bar{u} - k^2 u \bar{u}) dx,$$

where it is easy to see that the gradient integral makes sense (from elliptic estimates u is locally in $W^{2,p}$) hence

$$\Im \int_{\partial\Omega} u \frac{\partial \bar{u}}{\partial \nu} = \int_B \Im V |u|^2 \geq 0$$

We deduce that u has to be compactly supported and hence the unique continuation principle implies that $u = 0$. □

The far field pattern

Let start with the scattering interpretation of the Fourier transform.

Proposition

Let v be an outgoing solution of the inhomogeneous Helmholtz equation with a source f,

$$\Delta v + k^2 v = f,$$

where $f \in L^{2n/(n+2)}$ is compactly supported. Then v can be written when $|x| \to \infty$ as

$$v(x) = Ck^{(n-3)/2} \frac{e^{ik|x|}}{|x|^{(n-1)/2}} v_\infty(k, x/|x|) + o(|x|^{-(n-1)/2})$$

and the far field pattern is given by

$$v_\infty(k, x/|x|) = \hat{f}(kx/|x|) \tag{12}$$

Proof:

The proof follows from the volume potential formula

$$v(x) = \int \Phi_k(|x-y|)f(y)dy,$$

and the asymptotic expansion of the fundamental solution at $r = |x| \to \infty$
(this follows from asymptotic of Hankel functions + $|x-y| = |x| - \frac{x}{|x|} \cdot y + O(|x|^{-1})$)

$$\Phi_k(|x-y|) = C_n k^{(n-3)/2} e^{ikr} r^{-(n-1)/2} e^{-iky\cdot\frac{x}{r}} + O(r^{-n/2})) \quad (13)$$

□

Notice that the compactness of the support (y) is essential to obtain the above proposition.

If we apply proposition 1 to the Schrödinger equation, by taking $f = V(x)u(k,\omega,x)$, we obtain, since from estimates in lemma is a $L^{2n/(n+2)}$-function

Proposition

If V is compactly supported, then the far field pattern of the scattering solution is given by

$$u_\infty(k,\theta,\frac{x}{|x|}) = C\int_{\mathbf{R}^n} e^{-ikx/|x|\cdot y}V(y)u(k,\theta,y)dy. \tag{14}$$

For the case of non compactly supported potentials this is used as a definition of the scattering amplitude or far field pattern, see [ER]

To complete the proofs

1. Rellich Uniqueness theorem.
2. Unique continuation Principle.

Theorem

Let u be a function in the Sobolev space $W^{2,p}_{loc}$, for p such that $2/p - 1 = 1/r$ satisfying

$$|\Delta u(x)| \leq |V(x)u(x)| \tag{15}$$

in a domain Ω with $V \in L^r_{loc}$, where $r = n/2$ for $n \geq 3$ and $r > 1$ if $n = 2$. Then if u vanishes in an open subdomain of Ω it must vanish identically in Ω.

3. A priori estimates for the free resolvent.

Theorem

Let u be a function in the Sobolev space $W^{2,p}_{loc}$, for p such that $2/p - 1 = 1/r$ satisfying

$$|\Delta u(x)| \leq |V(x)u(x)| \tag{16}$$

in a domain Ω with $V \in L^r_{loc}$, where $r = n/2$ for $n \geq 3$ and $r > 1$ if $n = 2$. Then if u vanishes in an open subdomain of Ω it must vanish identically in Ω.

Remark 1 : The Theorem can be proved for V in the weak Lorentz space $L^{n/2,\infty}$ with sufficiently small norm. This result has been proved to be sharp [KN], [W].

Remark 2: There are also results for differential inequalities $|\Delta u(x)| \leq |V(x)u(x)| + |W(x) \cdot \nabla u(x)|$ The best result is due to T. Wolff and to Tataru.

The key point is the following Carleman estimate

The Carleman estimate

Proposition

Let $\rho \in \mathbf{R}$ and $v \in \mathbf{S}^{n-1}$, then there exists a $C > 0$ independent of ρ and v, such that for any $u \in \mathcal{C}_0^\infty$ if with p as above and q its dual exponent

$$\|e^{\rho v\cdot x}u\|_q \leq C|\rho|^{(1/p-1/q)n-2}\|e^{\rho v\cdot x}\Delta u\|_p \tag{17}$$

By taking $u = e^{-\rho v\cdot x}\tilde{u}$, then the estimate reduces to prove the a priori estimate

$$\|\tilde{u}\|_q \leq C|\rho|^{(1/p-1/q)n-2}\|(\Delta + 2\rho v\cdot\nabla + \rho^2)\tilde{u}\|_p.$$

Proof of unique continuation principle: We may reduce to the following
Claim: Suppose $u \in W^{2,p}_{loc}$ satisfies $|\Delta u(x)| \leq |V(x)u(x)|$ in a neighborhood of $\mathbf{S}^{n-1}$, where $V \in L^r_{loc}$. Then if u vanishes on one side of the sphere $\mathbf{S}^{n-1}$ it vanishes in a neighborhood of the sphere.

To obtain the theorem from the claim, consider the invariance by dilations and rotations of the statements and assume that $x_0 = 0$ is a point in the open set where u vanishes. Assume $d = dist(x_0, supp u \cap \Omega) < \infty$; by rescaling we may assume that $d = 1$, but from the claim u must vanish in a bigger ball, hence $d = \infty$.

To prove the claim let us assume first that $u = 0$ on the outside of $B(0,1)$. By dilation and rotation invariance assume that $u = 0$ outside of $B = B(-e_n, 1)$ where e_n is the n-th vector in the canonical basis of $\mathbf{R}^n$. It will be enough to prove that $u = 0$ in a neighborhood of the origin. Take $\eta \in \mathcal{C}_0^\infty([-2\delta, 2\delta])$, $\eta = 1$ on $[-\delta, \delta]$ such that

$$\|V\|_{L^r(A_1)} \leq \epsilon. \tag{18}$$

Then, from the Carleman estimate with $v = e_n$, denoting $\eta = \eta(x_n)$, $A_1 = B \cap \{x_n \geq -2\delta\}$ and $A_2 = B \cap \{-2\delta \leq x_n \leq -\delta\}$.

We have

$$\|e^{\rho x_n}\eta u\|_{L^q} \le C|\rho|^{(1/p-1/q)n-2}\|e^{\rho x_n}\Delta(\eta u)\|_{L^p}$$

$$\le C|\rho|^{(1/p-1/q)n-2}(\|e^{\rho x_n}u\Delta\eta\|_{L^p(A_2)}+\|e^{\rho x_n}\nabla\eta\cdot\nabla u\|_{L^p(A_2)}+\|e^{\rho x_n}\eta\Delta u\|_{L^p(}$$

$$\le C|\rho|^{(1/p-1/q)n-2}(e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)} + +C\|e^{\rho x_n}\eta V u\|_{L^p(A_1)}),$$

From Hölder

$$\le C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)}+C\epsilon|\rho|^{(1/p-1/q)n-2}\|e^{\rho x_n}\eta u\|_{L^q(A_1)}$$

take $C\epsilon < 1/2$ (actually condition (18) is not needed in dimension $n=2$, it is enough to take ρ sufficiently large), then

$$\|e^{\rho x_n}\eta u\|_{L^q} \le 2C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)}.$$

If we restrict to $A_3 == B \cap \{x_n \ge -\delta/2\}$ we have

$$e^{-\rho\delta/2}\|u\|_{L^q(A_3)} \le 2C|\rho|^{(1/p-1/q)n-2}e^{-\rho\delta}\|u\|_{W^{1,p}(A_1)}.$$

Hence by taking $\rho \to \infty$ we prove that

$$\|u\|_{L^q(A_3)} = 0.$$

Now let us assume that $u = 0$ in the interior of $B(0,1)$. By using the Kelvin transform we may reduce to the above case. In fact take

$$u_1(x) = u(x/|x|^2)|x|^{-(n-2)}.$$

Now notice that if u satisfies $\Delta u = W$, then $\Delta u_1 = W(x/|x|^2)|x|^{-n}$. Since $|W(x)| \leq |u(x)V(x)|$ we have

$$|\Delta u_1(x)| \leq |u(x/|x|^2)V(x/|x|^2)||x|^{-n} = |V_1(x)u_1(x)|,$$

where $V_1(x) = |x|^{-2}V(x/|x|^2)$. It follows that u_1 satisfies the conditions of the claim and then vanishes in the exterior of $B(0,1)$, hence u_1 vanishes in a neighborhood of $\mathbf{S}^{n-1}$. The same is true, then for u.

Calderón Problem

The problem of electrical impedance tomography deals with the reconstruction of the conductivity γ, in the potential equation on a bounded domain Ω, with the ellipticity condition $\gamma > c > 0$,

$$\operatorname{div} \gamma \nabla u = 0, \tag{19}$$

from boundary measurements given by yhe Dirichlet to Neumann (voltage-current) map at the boundary. One considers the Dirichlet boundary value problem

$$\begin{cases} \operatorname{div} \gamma \nabla u = 0 \ \text{ in } \Omega \\ u_{|\partial\Omega} = f. \end{cases} \tag{20}$$

The D-N map is given by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu} \tag{21}$$

where the last denotes the normal derivative at a boundary point. This problem under regularity is reduced to the similar problem for the Schrödinger equation.

Reduction to the Schrödinger equation.

Assume $\gamma \in \mathcal{C}^2(\bar{\Omega})$.

Lemma

$$\operatorname{div}(\gamma \nabla u) = \gamma^{1/2}(\Delta + q)(\gamma^{1/2} u), \tag{22}$$

where

$$q = -\frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}. \tag{23}$$

The change $v = \gamma^{1/2} u$. reduces the conductivity equation to the Schrödinger equation.

The inverse BVP for Schrödinger equation

Let Ω be a bounded domain, which we assume smooth, we consider the Schrödinger hamiltonian $\Delta + V$ with the electrostatic potential q, which we assume in $L^r(\Omega)$. We try to recover q from boundary measurements. The measurements are given by Dirichlet to Neumann map (D-N map). For a given boundary Dirichlet datum $f \in W^{1/2,2}(\partial\Omega)$ its image by the D-N map is defined as

$$\Lambda_q(f) = \frac{\partial}{\partial \nu} u \tag{24}$$

where u is the solution of the problem

$$\begin{cases} (\Delta + q)u = 0 \ \text{ in } \Omega \\ u_{|\partial\Omega} = f. \end{cases} \tag{25}$$

Since 0 could be an eigenvalue of the Dirichlet operator, the uniqueness of the above problem can not in general be proved and hence the D-N not need to be a map. In order to avoid this constrain, we are going to substitute the D-N map by the so called "Cauchy data set" of q which is defined by

$$\mathcal{C}_q = \{(u_{|\partial\Omega}, \frac{\partial}{\partial\nu}u) : u \in W^{1,2}(\bar{\Omega}), (\Delta + q)u = 0\} \qquad (26)$$

Remark that in the case that the D-N map exists, $\mathcal{C}_q$ is a graph. In general we may claim

Proposition

Let $q \in L^r$, with $r \geq n/2$, then

$$\mathcal{C}_q \subset W^{1/2,2}(\partial\Omega) \times W^{-1/2,2}(\partial\Omega) \qquad (27)$$

The Cauchy data set

Proof: the problem is that, a priori, with the only assumption $u \in W^{1,2}(\bar{\Omega})$, we can not claim that the trace of $\bigtriangledown u$ is in $W^{-1/2,2}(\partial\Omega)$, but, by using the equation, we are going to prove that $\frac{\partial}{\partial\nu} u \in W^{-1/2,2}(\partial\Omega)$.
Assume that u is a weak $W^{1,2}$ solution of $(\Delta + q)u = 0$. This means that for any $\psi \in \mathcal{C}_0^\infty(\Omega)$ one has

$$-\int_\Omega \bigtriangledown u \cdot \bigtriangledown \psi + \int_\Omega q u \psi = 0 \tag{28}$$

Assume that u is a smooth solution of the equation and take a test funtion $\phi \in W^{1/2,2}(\partial\Omega)$, by the trace theorem we can extend ϕ to a $W^{1,2}(\Omega)$ function $\tilde{\phi}$ such that

$$\|\tilde{\phi}\|_{W^{1,2}(\Omega)} \leq C\|\phi\|_{W^{1/2,2}(\partial\Omega)}.$$

By the Green formula

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma = \int_{\Omega} (\nabla u \cdot \nabla\tilde{\phi}) + \int_{\Omega} \Delta u\tilde{\phi} \\ = \int_{\Omega} (\nabla u \cdot \nabla\tilde{\phi}) - \int_{\Omega} (qu\tilde{\phi}) \tag{29}$$

Hence by Hölder inequality

$$|\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma| \leq \| \nabla u\|_2 \| \nabla \tilde{\phi}\|_2 + \|q\|_r \|u\|_{p'} \|\tilde{\phi}\|_{p'},$$

where $1/r + 1/p' + 1/p' = 1$ and $1/p - 1/p' = 1/r$.
Since we have $1/r = 1 - 2/p' \leq 2/n$, then $1/2 - 1/p' \leq 1/n$ and we can use the Sobolev embedding, $\|\tilde{\phi}\|_{p'} \leq C\|\tilde{\phi}\|_{W^{1,2}(\Omega)}$, which together with the trace estimate give us:

$$|\int_{\partial\Omega} \frac{\partial}{\partial\nu} u\phi d\sigma| \leq C\| u\|_{W^{1,2}} \|\phi\|_{W^{1/2,2}(\partial\Omega)}.$$

This proves that the normal derivative can be defined, from (29) by density, as an element of $W^{-1/2,2}(\partial\Omega)$, with the only assumption $u \in W^{1,2}(\Omega)$.

The change $v = \gamma^{1/2}u$ reduces the conductivity equation to the Schrödinger equation with $q = -\frac{\Delta(\gamma^{1/2})}{\gamma^{1/2}}$. To achieve this we need the following relation between maps and Cauchy data sets

Lemma

Let us denote

$$g = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2}f) + 1/2\gamma^{-1}\frac{\partial\gamma}{\partial\nu}f. \tag{30}$$

Then (f, g) are Cauchy data for the associated Schrödinger equation.

We need the recovery of boundary values of the conductivity γ and its normal derivative from Λ_γ (Kohn-Vogelius, Sylvester-Uhlmann, Alessandrini, Brown).

Uniqueness of the inverse problem

Now we can state the main result of this section, the uniquenes of the inverse boundary value problem. This is a L^r-version of Sylvester and Uhlmann's pioneering result due to Jerison and Kenig and Chanillo.

Theorem

Let q_1 and q_2 be functions in $L^r(\Omega)$, $r > n/2$, $n \geq 3$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then $q_1 = q_2$.

The proof is based on the existence of Calderón approximated solutions :

Calderón-Sylvester-Uhlmann solutions

Proposition (Sylvester-Uhlmann solutions)

Let $\rho \in \mathbf{C}^n$ *such that* $\rho \cdot \rho = 0$ *and* $q = q_1 \chi_\Omega$ *with* $q_1 \in L^r$, $r \geq n/2$ *and* $\|q_1\|_{n/2} \leq \epsilon(n)$ *if* $r = n/2$. *Then for* $|\rho|$ *sufficiently large there exists a* $W^{1.2}_{loc}$ *solution* u *of* $(\Delta + q)u = 0$ *in* $\mathbf{R}^n$ *which can be written as*

$$u(x) = e^{\rho \cdot x}(1 + \psi(\rho, x)),$$

where for $r > n/2$

$$\|\psi(\rho, \cdot)\|_{p'} \to 0 \text{ as } |\rho| \to \infty. \tag{31}$$

Proof:

Insert $u = e^{\rho\cdot x}(1+\psi(\rho,x))$ in the equation, then we are reduced to find a solution of the Faddeev equation

$$(\Delta + 2\rho\cdot\nabla)\psi = q + q\psi, \tag{32}$$

satisfying (31). If we take Fourier transform in (32), we are reduced to find a solution of the integral equation

$$\psi = K_\rho(q) + K_\rho(q\psi), \tag{33}$$

where $\widehat{K_\rho(f)}(\xi) = (-|\xi|^2 - 2i\rho\cdot\xi)^{-1}\hat{f}(\xi)$.
The mapping properties of K_ρ we use are

$$\|K_\rho(q)\|_{p'} \leq |\rho|^{n/r-2}\|q\|_p.$$

This is the a priori estimate for
$2/(n+1) < 1/p - 1/p' = 1/r \leq 2/n$

$$\|f\|_{p'} \leq C|\rho|^{n/r-2}\|(\Delta + 2\rho\cdot\nabla)f\|_p.$$

We write $T_\rho(f) = K_\rho(qf)$. Then we have to solve the Fredholm equation

$$(I - T_\rho)(\psi) = K_\rho(q).$$

This and Hölder inequality give

$$\|T_\rho(f)\|_{p'} \leq C\|qf\|_p \leq C\|q\|_r\|f\|_{p'},$$

from the assumptions on q we have that T_ρ is bounded in $L^{p'}$ with norm less than 1. Hence we can write

$$\psi = (I - T_\rho)^{-1} K_\rho(q)$$

and

$$\|\psi\|_{p'} \leq \|K_\rho(q)\|_{p'} \leq |\rho|^{n/r-2}\|q\|_p.$$

Since $p < r$ this gives condition (31) if $r > n/2$. The fact that $u \in W^{1,2}$ follows from a priori estimates for the Laplace operator since $\Delta u = qu \in L^p$ and is compactly supported.

We state the other ingredient in the proof of theorem 8.

Proposition

Let $q_i \in L^r(\Omega)$, $i = 1, 2$ and $r \geq n/2$ such that $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, and assume that u_i are $W^{1,2}(\bar{\Omega})$-solutions of $(\Delta + q_i)u_i = 0$ in Ω. Then

$$\int_{\Omega} (q_1 - q_2) u_1 u_2 = 0. \tag{34}$$

Proof: Let $(f_i, g_i) \in \mathcal{C}_{q_i}$ be the Cauchy data generated by u_i. From the fact that the Cauchy data for both potentials coincide, there exists a function $v_1 \in W^{1,2}(\bar{\Omega})$ satisfying $(\Delta + q_1)v_1 = 0$ with Cauchy data (f_2, g_2). Hence

$$0 = g_2(f_1) - g_2(f_1) = \frac{\partial}{\partial \nu} v_1(f_1) - \frac{\partial}{\partial \nu} u_2(f_1)$$

From (29) we have

$$\frac{\partial}{\partial \nu} v_1(f_1) = \int_{\Omega} \nabla v_1 \cdot \nabla \tilde{f}_1 - \int_{\Omega} q_1 v_1 \tilde{f}_1$$

We may choose the extension $\tilde{f}_1 = u_1$, hence

$$\frac{\partial}{\partial \nu} v_1(f_1) = \int_\Omega \bigtriangledown v_1 \cdot \bigtriangledown u_1 - \int_\Omega q_1 v_1 u_1$$

$$= \frac{\partial}{\partial \nu} u_1(f_2)$$

since v_1 is an extension of f_2. Then we have

$$0 = \frac{\partial}{\partial \nu} u_1(f_2) - \frac{\partial}{\partial \nu} u_2(f_1)$$

$$= \int_\Omega \bigtriangledown u_1 \cdot \bigtriangledown u_2 - \int_\Omega q_1 u_1 u_2 - \int_\Omega \bigtriangledown u_2 \cdot \bigtriangledown u_1 + \int_\Omega q_2 u_2 u_1$$

This proves the proposition.

Proof of uniqueness theorem :

Fix $\xi \in \mathbf{R}^n$ and take the two complex vectors $\rho_1 = l + i(\xi + m)$ and $\rho_2 = -l + i(\xi - m)$, where l, m and ξ are vectors in $\mathbf{R}^n$ orthogonal to each other and $|l| = |\xi + m|$. From proposition 5 we can take for $i = 1, 2$ a solution of $(\Delta + q_i)u_i = 0$ of the form $u_i = e^{\rho_i \cdot x}(1 + \psi_i(\rho, x))$. Then from proposition 25 we have

$$0 = \int_\Omega (q_1 - q_2) u_1 u_2 = \int_\Omega (q_1 - q_2) e^{\rho_1 \cdot x}(1 + \psi_1(\rho, x)) e^{\rho_2 \cdot x}(1 + \psi_2(\rho, x)) dx$$

$$= \int_\Omega (q_1 - q_2) e^{2i\xi \cdot x} dx + \int_\Omega (q_1 - q_2) e^{2i\xi \cdot x} \psi_1(\rho_1, x)(1 + \psi_2(\rho_2, x))$$
$$+ \int_\Omega (q_1 - q_2) e^{2i\xi \cdot x} \psi_2(\rho_2, x)(1 + \psi_1(\rho_1, x)).$$

But

$$\left| \int_\Omega q_i e^{2i\xi \cdot x} \psi_2(\rho_2, x)(1 + \psi_1(\rho_1, x)) \right| \leq \|q_i\|_r \|\psi_2\|_{p'} \|1 + \psi_2\|_{L^{p'}(\Omega)}$$

tends to zero as $|\rho| = c|l|$ tends to ∞, hence we obtain $\hat{q}_1 - \hat{q}_2 = 0$.

- The proof can be extended for potentials in $L^{n/2}$. In this case one proves that in proposition 5 $\|\psi(\rho, \cdot)\|_{p'}$ tends to zero in the weak sense, which is what we need to apply proposition 6 There is also an extension for potential in Morrey spaces, see [Ch] which is based on the uniform Sobolev estimates of [ChS] and [ChiR], this result contains potential in the Lorentz space $L^{n/2,\infty}$ with small norm.
- The 2D case for the Schrodinger equation was solved by Bukhgeim, in the case of $q \in L^\infty(\Omega)$. For the special case of potential coming from conductivities with two derivatives it was proved by Nachman, and with just one derivative by Brown and Uhlmann. They use the scattering transform of the potential. Let us remark that, in this case, the inverse problem is formally well determined and one needs to control all the solutions of Faddeev equation, even for $|\rho| = 0$.
- Special Riemannian manifolds (wedge products).
- As in the case of the scattering problem, there is a similar fix energy Calderon problem which can be seen as the above for $q(x) = V(x) - k^2$ where k is fixed, without any change.

Back to Calderón problem

- Regularity: It requires ($n \geq 3$) to reduce to Schrödinger equation: Sylverter-Uhlmann, Brown ($\gamma \in W^{3/2}$) Haberman-Tataru ($\mathcal{C}^1$, small Lipschitz norm) Haberman ($W^{1,n}$, $n = 3, 4$), Caro-Roger (Lipschitz $n > 4$). New estimates.
- 2d: Calderón conjecture was solved by Astala and Päivärinta, by using Beltrami equation.
- Other Schrödinger equations. Magnetic potentials. Other integral transform.
- Partial data problems and local data problem.

Summary

Estimates needed:

- Resolvent $R_+(k^2)(f) = (\Delta + k^2 + i0)^{-1}(f)$,

$$\|R_+(k^2)f\|_{W^{s,p'}(<x>^{-\beta}dx)} \leq C(k)\|f\|_{L^p(<x>^{\beta}dx)}.$$

- Carleman $(\rho \in \mathbb{R})$

$$\|f\|_q \leq C|\rho|^{(1/p-1/q)n-2}\|(\Delta + 2\rho v \cdot \bigtriangledown + \rho^2)f\|_p.$$

- Fadeev $(\rho \in \mathbb{C}^n)$

$$\|f\|_{p'} \leq C|\rho|^{n/r-2}\|(\Delta + 2\rho \cdot \bigtriangledown)f\|_p.$$

Summary

Estimates needed:

- Resolvent L.A.P: limit values as $0 < \Im z \to 0$ of

$$\|f\|_{W^{s,p'}(<x>^{-\beta}dx)} \leq C(z)\|(\Delta + z)f\|_{L^p(<x>^{\beta}dx)}.$$

- Carleman: $(\rho \in \mathbb{R})$

$$\|f\|_q \leq C|\rho|^{(1/p-1/q)n-2}\|(\Delta + 2\rho v \cdot \bigtriangledown + \rho^2)f\|_p.$$

- Fadeev: $(\rho \in \mathbb{C}^n,\ \rho \cdot \rho = 0)$

$$\|f\|_{p'} \leq C|\rho|^{n/r-2}\|(\Delta + 2\rho \cdot \bigtriangledown)f\|_p.$$

Uniform Sobolev estimate

$$\|f\|_{W^{s,p'}(<x>^{-\beta}dx)} \leq C(\rho, z)\|(\Delta + \rho \cdot \nabla + z)f\|_{L^p(<x>^{\beta}dx)}.$$